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# PARTITIONS INTO SPECIFIED PARTS WHICH APPEAR A SPECIFIED NUMBER OF TIMES

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ABSTRACT. Restrictions of this type, which are known to produce partition identities when any natural number may be used as a part, are shown to produce partition identities when only certain parts may be used.

1. **Introduction.** Let $N$ be the set of natural numbers and let $A$, $B$, and $S$ be subsets of $N$. $S(n)$ is the number of partitions of $n$ into parts which are elements of $S$. $A^*(n, B)$ is the number of partitions of $n$ into parts which are elements of $B$, such that each part appears exactly $m$ times for some $m \in A$. For example, if $1, 2 \in A$ and $3, 4, 5 \in B$, then $5 + 4 + 3 + 3$ is a partition of 15 of the type enumerated by $A^*(15, B)$.

$A$ is *admissible applied to* $B$ if there exists an $S$ such that $A^*(n, B) = S(n)$ for every $n \in N$. In this case, $S$ corresponds to $A$. For example, previous results by this author [6] state that $A$ is admissible applied to $N$ if it is one of the following, where $k, d \in N$:

$A = \{m \mid m < (2d-1)k$ and $m \not\equiv (2i-1)k + j$ with $1 \leq i \leq d-1,\ 0 \leq j \leq k-1\}$, or

$A = \{m \mid m \not\equiv (2i-1)k + j$ with $1 \leq i \leq d,\ 0 \leq j \leq k-1\}$, or

$A = \{m \mid m \not\equiv (2i-1)k + j$ with $1 \leq i,\ 0 \leq j \leq k-1\}$.

The purpose of this paper is to show that these sets are admissible applied to $B$ for $B \not\equiv N$. In particular, for each set $A$, there is a class of sets such that $A$ is admissible applied to $B$ for every $B$ in this class. In each case, $N$ is a member of the class and, thus, these results generalize the earlier ones.

2. **Results.**

**Theorem 1.** *Let $k, d \in N$, $B \subseteq N$ such that*

$$kB \cap 2dkB \subseteq B \cap 2kB \subseteq kB \cup 2dkB \subseteq B \cup 2kB.$$

*Then the number of partitions of $n$ into elements of $B$, such that no part appears exactly $(2i-1)k + j$ times with $1 \leq i \leq d-1$, $0 \leq j \leq k-1$ or more than $(2d-1)k - 1$ times, is equal to the number of partitions of $n$ into elements of*

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$$S = [(B \cup 2kB) - (kB \cup 2dkB)] \cup [(B \cap 2kB) - (kB \cap 2dkB)].$$

A result of Glaisher [4], states that for $k \in N$, the set $A = \{i\}_{i=1}^{k-1}$ is admissible applied to $N$. Letting $d = 1$ in Theorem 1, we see that this set is admissible applied to $B$ for every $B$ such that $kB \subseteq B$.

**Corollary 1 (Subbarao).** *Let $k \in N$, $B \subseteq N$ such that $kB \subseteq B$. Then the number of partitions of $n$ into elements of $B$, such that parts appear at most $k-1$ times, is equal to the number of partitions of $n$ into elements of $S = B - kB$.*

Letting $k = 1$ in Theorem 1, we obtain a similar result.

**Corollary 2.** *Let $d \in N$, $B \subseteq N$ such that $dB \subseteq B$. Then the number of partitions of $n$ into elements of $B$, such that parts appear exactly $2i$ times with $1 \le i \le d-1$, is equal to the number of partitions of $n$ into elements of $S = 2[B - dB]$.*

When $B = \{id^j | j \ge 0,\ 1 \le i \le d-1\}$ in Corollary 2, $A$ is not only admissible applied to $B$, but the corresponding set is $A$ itself.

**Corollary 3.** *Let $d \in N$, $B = \{id^j | j \ge 0,\ 1 \le i \le d-1\}$. Then the number of partitions of $n$ into elements of $B$, such that each part appears exactly $2i$ times with $1 \le i \le d-1$, is equal to the number of partitions of $n$ into parts of the form $2i$ with $1 \le i \le d-1$.*

**Theorem 2.** *Let $k$, $d \in N$, $B \subseteq N$ such that*

$$\begin{aligned} B \cap 2kB \cap (2d+1)kB &\subseteq kB \cap (4d+2)kB \subseteq (B \cap 2kB) \\ &\cup (B \cap (2d+1)kB) \cup (2kB \cap (2d+1)kB) \subseteq kB \\ &\cup (4d+2)kB \subseteq B \cup 2kB \cup (2d+1)kB. \end{aligned}$$

*Then the number of partitions of $n$ into elements of $B$, such that no part appears exactly $(2i-1)k + j$ times with $1 \le i \le d$, $0 \le j \le k-1$, is equal to the number of partitions of $n$ into elements of*

$$\begin{aligned} S = {} & [(B \cup 2kB \cup (2d+1)kB) - (kB \cup (4d+2)kB)] \\ & \cup [((B \cap 2kB) \cup (B \cap (2d+1)kB) \cup (2kB \cap (2d+1)kB)) \\ & \qquad - (kB \cap (4d+2)kB)] \\ & \cup [B \cap 2kB \cap (2d+1)kB]. \end{aligned}$$

**Corollary 4.** *Let $d$, $u \in N$, $B = \{2^i | i \ge u-1\}$. Then the number of partitions of $n$ into elements of $B$, where no part appears exactly $2i-1$ times with $1 \le i \le d$, is equal to the number of partitions of $n$ into elements of $S = \{2^i | i \ge u\} \cup \{(2d+1)2^{u-1}\}$.*

**Theorem 3.** *Let $k \in N$, $B \subseteq N$ such that $B \cap 2kB \subseteq kB \subseteq B \cup 2kB$. Then the number of partitions of $n$ into elements of $B$, where no part appears exactly $(2i-1)k+j$ times with $1 \le i$, $0 \le j \le k-1$, is equal to the number of partitions of $n$ into elements of $S = [(B \cup 2kB) - kB] \cup [B \cap 2kB]$.*

In conclusion, we note that, for all $k, d, m \in N$, the set $B = \{im\}_{i=1}^{\infty}$ satisfies the requirements of the three theorems of this paper.

## 3. Proofs.

**Proof of Theorem 1.** The generating function for $A^*(n, B)$ is

$$\prod_{n \in B}\left[\frac{1}{1-x^n} - \sum_{i=1}^{d-1}\sum_{j=0}^{k-1} x^{((2i-1)k+j)n} - \sum_{j=(2d-1)k}^{\infty} x^{jn}\right]$$

$$= \prod_{n \in B}\left[\frac{1}{1-x^n} - \left(\sum_{i=1}^{d-1} x^{(2i-1)kn}\right)\left(\sum_{j=0}^{k-1} x^{jn}\right) - \sum_{j=(2d-1)k}^{\infty} x^{jn}\right]$$

$$= \prod_{n \in B}\left[\frac{1}{1-x^n} - x^{kn}\left(\frac{1-x^{(2d-2)kn}}{1-x^{2kn}}\right)\left(\frac{1-x^{kn}}{1-x^n}\right) - \frac{x^{(2d-1)kn}}{1-x^n}\right]$$

$$= \prod_{n \in B}\left(\frac{1}{1-x^n}\right)\left[1 - x^{kn}\left(\frac{1-x^{(2d-2)kn}}{1+x^{kn}}\right) - x^{(2d-1)kn}\right]$$

$$= \prod_{n \in B}\left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)[(1+x^{kn}) - x^{kn}(1-x^{(2d-2)kn}) - x^{(2d-1)kn}(1+x^{kn})]$$

$$= \prod_{n \in B}\left(\frac{1}{1-x^n}\right)\left(\frac{1}{1+x^{kn}}\right)[1-x^{2dkn}]$$

$$= \prod_{n \in B}\left(\frac{1}{1-x^n}\right)\left(\frac{1-x^{kn}}{1-x^{2kn}}\right)(1-x^{2dkn}).$$

Now $kB \cup 2dkB \subseteq B \cup 2kB$. So terms of the forms $1-x^{kn}$ and $1-x^{2dkn}$ with $n \in B$, cancel with terms of the form $1-x^m$ or $1-x^{2km}$ with $m \in B$. In fact, $kB \cap 2dkB \subseteq B \cap 2kB$. So, if for some $n, n' \in B$, $1-x^{kn} = 1-x^{2dkn'}$, then both terms cancel with terms $1-x^m = 1-x^{2km'}$ with $m, m' \in B$. Also, $B \cap 2kB \subseteq kB \cup 2dkB$. So, if $1-x^n = 1-x^{2kn'}$ with $n, n' \in B$, then at least one of these terms cancels with a term of the form $1-x^{km}$ or $1-x^{2dkm}$ with $m \in B$. Hence, this is the generating function for $S(n)$ for some $S$.

The elements of $N$ which are in $S$ are those in $(B \cup 2kB) - (kB \cup 2dkB)$ or $(B \cap 2kB) - (kB \cap 2dkB)$.

**Proof of Corollary 1.** When $d = 1$, $2dkB = 2kB$, and the restriction on $B$ in Theorem 1 is

$$kB \cap 2kB \subseteq B \cap 2kB \subseteq kB \cup 2kB \subseteq B \cup 2kB,$$

which is clearly satisfied if $kB \subseteq B$.

When $d = 1$, there is no $i$ such that $1 \leq i \leq d - 1$, so parts can appear at most $(2d - 1)k - 1 = k - 1$ times.

Also,

$$\begin{aligned} S &= [(B \cup 2kB) - (kB \cup 2kB)] \cup [(B \cap 2kB) - (kB \cap 2kB)] \\ &= [(B - kB) - ((B - kB) \cap 2kB)] \cup [(B - kB) \cap 2kB] \\ &= B - kB. \end{aligned}$$

**Proof of Corollary 2.** When $k = 1$, $kB = B$ and $2dkB = 2dB$. So, the restriction on $B$ in Theorem 1 is

$$B \cap 2dB \subseteq B \cap 2B \subseteq B \cup 2dB \subseteq B \cup 2B,$$

which is clearly satisfied if $2dB \subseteq 2B$; i.e., $dB \subseteq B$.

When $k = 1$, no part can appear $2i - 1$ times with $1 \leq i \leq d - 1$ or more than $2d - 2$ times. So parts can appear $2i$ times with $1 \leq i \leq d - 1$.

Also,

$$\begin{aligned} S &= [(B \cup 2B) - (B \cup 2dB)] \cup [(B \cap 2B) - (B \cap 2dB)] \\ &= [(2B - 2dB) - ((2B - 2dB) \cap B)] \cup [(2B - 2dB) \cap B] \\ &= 2B - 2dB = 2[B - dB]. \end{aligned}$$

**Proof of Corollary 3.** When $B = \{id^j | j \geq 0,\ 1 \leq i \leq d - 1\}$, $dB = \{id^j | j \geq 1,\ 1 \leq i \leq d - 1\} \subseteq B$.

Also,

$$\begin{aligned} S &= 2[B - dB] = 2[\{id^j | j = 0,\ 1 \leq i \leq d - 1\}] \\ &= \{2i | 1 \leq i \leq d - 1\} = A. \end{aligned}$$

**Proof of Theorem 2.** The generating function for $A^*(n, B)$ is

$$\prod_{n \in B} \left[ \frac{1}{1 - x^n} - \sum_{i=1}^{d} \sum_{j=0}^{k-1} x^{((2i-1)k+j)n} \right]$$

$$= \prod_{n \in B} \left[ \frac{1}{1 - x^n} - \left( \sum_{i=1}^{d} x^{(2i-1)kn} \right) \left( \sum_{j=0}^{k-1} x^{jn} \right) \right]$$

$$= \prod_{n \in B} \left[ \frac{1}{1 - x^n} - x^{kn} \left( \frac{1 - x^{2dkn}}{1 - x^{2kn}} \right) \left( \frac{1 - x^{kn}}{1 - x^n} \right) \right]$$

$$= \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left[ 1 - x^{kn} \left( \frac{1 - x^{2dkn}}{1 + x^{kn}} \right) \right]$$

$$= \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left( \frac{1}{1 + x^{kn}} \right) [(1 + x^{kn}) - x^{kn}(1 - x^{2dkn})]$$

$$= \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left( \frac{1}{1 + x^{kn}} \right) [1 + x^{(2d+1)kn}]$$

$$= \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left( \frac{1 - x^{kn}}{1 - x^{2kn}} \right) \left( \frac{1 - x^{(4d+2)kn}}{1 - x^{(2d+1)kn}} \right).$$

Now $kB \cup (4d + 2)kB \subseteq B \cup 2kB \cup (2d + 1)kB$. So terms of the forms $1 - x^{kn}$ and $1 - x^{(4d+2)kn}$, with $n \in B$, cancel with terms of the form $1 - x^m$ or $1 - x^{2km}$ or $1 - x^{(2d+1)km}$ with $m \in B$. In fact,

$kB \cap (4d + 2)kB \subseteq (B \cap 2kB) \cup (B \cap (2d + 1)kB) \cup (2kB \cap (2d + 1)kB)$.

So, if $1 - x^{kn} = 1 - x^{(4d+2)kn'}$ with $n, n' \in B$, then both terms cancel with terms $1 - x^m = 1 - x^{2km'}$ or $1 - x^m = 1 - x^{(2d+1)km'}$ or $1 - x^{2km} = 1 - x^{(2d+1)km'}$ with $m, m' \in B$.

$(B \cap 2kB) \cup (B \cap (2d + 1)kB) \cup (2kB \cap (2d + 1)kB) \subseteq kB \cup (4d + 2)kB$.

So, if $1 - x^n = 1 - x^{2kn'}$ or $1 - x^n = 1 - x^{(2d+1)kn'}$ or $1 - x^{2kn} = 1 - x^{(2d+1)kn'}$ with $n, n' \in B$, then at least one of these terms cancels with a term of the form $1 - x^{km}$ or $1 - x^{(4d+2)km}$ with $m \in B$.

Additionally, $B \cap 2kB \cap (2d + 1)kB \subseteq kB \cap (4d + 2)kB$. So, if $1 - x^n = 1 - x^{2kn'} = 1 - x^{(2d+1)kn''}$ with $n, n', n'' \in B$, then two of these terms cancel with terms $1 - x^{km} = 1 - x^{(4d+2)km'}$ with $m, m' \in B$. Hence, this is the generating function for $S(n)$ for some $S$.

The elements of $N$ which are in $S$ are those in

$$(B \cup 2kB \cup (2d + 1)kB) - (kB \cup (4d + 2)kB)$$

or in

$$[(B \cap 2kB) \cup (B \cap (2d + 1)kB) \cup (2kB \cap (2d + 1)kB)] - [kB \cap (4d + 2)kB]$$

or in $B \cap 2kB \cap (2d + 1)kB$.

**Proof of Corollary 4.** When $k = 1$ and $B = \{2^i \mid i \geq u - 1\}$, $kB = B$, $2kB = 2B = \{2^i \mid i \geq u\}$, $(2d + 1)kB = (2d + 1)B = \{(2d + 1)2^i \mid i \geq u - 1\}$, and $(4d + 2)kB = (4d + 2)B = \{(2d + 1)2^i \mid i \geq u\}$. Thus, $B \cap (2d + 1)kB = \emptyset$, $2kB \cap (2d + 1)kB = \emptyset$, $kB \cap (4d + 2)kB = \emptyset$, and $2kB \subseteq B$. Hence, the condition imposed on $B$ in Theorem 2 is

$$\emptyset \subseteq \emptyset \subseteq 2B \subseteq B \cup (4d+2)B \subseteq B \cup (2d+1)B,$$

which is clearly satisfied.

Also,

$$\begin{aligned} S &= [(B \cup 2kB \cup (2d+1)kB) - (kB \cup (4d+2)kB)] \\ &\quad \cup [((B \cap 2kB) \cup (B \cap (2d+1)kB) \\ &\qquad\qquad \cup (2kB \cap (2d+1)kB)) - (kB \cap (4d+2)kB)] \\ &\quad \cup [B \cap 2kB \cap (2d+1)kB] \\ &= [(B \cup (2d+1)B) - (B \cup (4d+2)B)] \cup [2B - \emptyset] \cup [\emptyset] \\ &= [(2d+1)B - (4d+2)B] \cup 2B \\ &= \{(2d+1)2^i \mid i = u-1\} \cup \{2^i \mid i \geq u\}. \end{aligned}$$

**Proof of Theorem 3.** The generating function for $A^*(n, B)$ is

$$\begin{aligned} &\prod_{n \in B} \left[ \frac{1}{1-x^n} - \sum_{i=1}^{\infty} \sum_{j=0}^{k-1} x^{((2i-1)k+j)n} \right] \\ &= \prod_{n \in B} \left[ \frac{1}{1-x^n} - \left( \sum_{i=1}^{\infty} x^{(2i-1)kn} \right) \left( \sum_{j=0}^{k-1} x^{jn} \right) \right] \\ &= \prod_{n \in B} \left[ \frac{1}{1-x^n} - x^{kn} \left( \frac{1}{1-x^{2kn}} \right) \left( \frac{1-x^{kn}}{1-x^n} \right) \right] \\ &= \prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left[ 1 - x^{kn} \left( \frac{1}{1+x^{kn}} \right) \right] \\ &= \prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left( \frac{1}{1+x^{kn}} \right) = \prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left( \frac{1-x^{kn}}{1-x^{2kn}} \right). \end{aligned}$$

Now, $kB \subseteq B \cup 2kB$. So, terms of the form $1 - x^{kn}$ with $n \in B$, cancel with terms of the form $1 - x^m$ or $1 - x^{2km}$ with $m \in B$. Also, $B \cap 2kB \subseteq kB$. So, if $1 - x^n = 1 - x^{2kn'}$ with $n, n' \in B$, then one of these terms cancels with a term of the form $1 - x^{km}$ with $m \in B$. Hence, this is the generating function for $S(n)$ for some $S$.

The elements of $N$ which are in $S$ are those in $(B \cup 2kB) - kB$ or in $B \cap 2kB$.

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