SUM OF A DOUBLE SERIES

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ABSTRACT. In this paper we obtain the sum of a double series $F(1, 1)$ and, in a particular case, we get a new formula

$$\frac{\Gamma(\beta - \rho - a) \Gamma(\beta)}{\Gamma(\beta - \rho) \Gamma(\beta - a)} \left( \begin{array}{c} a, \beta - a, \frac{1}{2} \rho + \frac{1}{2}; 1 \\ \frac{1}{2} + \frac{1}{2} \beta, \frac{1}{2}(1 + \beta), 1 + \rho \end{array} \right)$$

provided that $R(\beta - a) > 0$, $R(\beta - \rho - a) > 0$ and $R(\beta - \rho) > 0$. If $a = -n$, the formula reduces to a known result due to Bailey [2].


Professor Carlitz' papers created interest in the summation formulae, because these formulae have applications in the solutions of certain problems in theoretical physics. Following Professor Carlitz' papers, Jain [6] and Sharma [8] have proved some new summation formulae for double series. The object of this paper is to give the sum of a certain double hypergeometric series, and on specializing the parameters, we get a result of Bailey [2, (8.2)]. The results obtained in this paper are believed to be new.

The following notation of Chaundy [3] will be used to represent the hypergeometric series of higher order and of two variables.

$$F\left( \begin{array}{c} (a_p); (b_q); (c_r); x, y \\ (d_s); (e_t); (f_k); \end{array} \right) = \sum_{m,n=0}^{\infty} \frac{[(a_p)]_{m+n} [(b_q)]_m [(c_r)]_n x^m y^n}{[(d_s)]_{m+n} [(e_t)]_m [(f_k)]_n m! n!},$$

where $(a_p)$ and $[(a_p)]_{m+n}$ will mean $a_1, \ldots, a_p$ and $(a_1)_{m+n}, \ldots, (a_p)_{m+n}$.

In the investigation we require Slater's formula [7, p. 65, (2.4.2.2)]:

$$\frac{\Gamma(1 + f - g, \frac{1}{2} f + \frac{1}{2}); 1}{\Gamma(\frac{1}{2} + \frac{1}{2} f - \frac{1}{2} g - \frac{1}{2} n, 1 + \frac{1}{2} f - \frac{1}{2} g - \frac{1}{2} n)} = \frac{(g)_n}{(g - f)_n}.$$
valid for $R(\beta) > 0$, $R(\beta - \alpha) > 0$, $R(\beta + f_1 + f_2) > 0$ and $R(\beta + f_1 + f_2 - \alpha) > 0$.

Proof. To prove (3), we consider

$$F[\alpha, \beta + f_1 + f_2 - \alpha; \frac{1}{2}f_1, \frac{1}{2} + \frac{1}{2}f_1; \frac{1}{2}f_2, \frac{1}{2} + \frac{1}{2}f_2; 1, 1]$$

$$= \frac{\Gamma(\beta + f_1 + f_2 - \alpha)}{\Gamma(\beta)\Gamma(\beta + f_1 + f_2)}$$

This completes the proof of (3) under the condition stated therein. We shall mention some of the interesting particular cases of (3).

(a) In case $\alpha = -n$ (a positive integer) in (3), we get
\[
F\left[ \begin{array}{c}
-n, \beta + f_1 + f_2 + n; \frac{1}{2}f_1, \frac{1}{2} + \frac{1}{2}f_1; \frac{1}{2}f_2, \frac{1}{2} + \frac{1}{2}f_2; 1, 1 \\
\frac{1}{2}(\beta + f_1 + f_2), \frac{1}{2}(1 + \beta + f_1 + f_2); 1 + f_1; 1 + f_2;
\end{array} \right] = (\beta)^n/(\beta + f_1 + f_2)^n.
\]

In case \( f_2 = 0 \) in (5), it reduces to a known result of Bailey [2, (8.2)].

(b) In case \( f_2 = 0 \) in (3), we get a new summation formula for \(_4F_3(1)\):

\[
\begin{align*}
_4F_3\left[ \begin{array}{c}
a, \beta - \alpha, \frac{1}{2}p, \frac{1}{2}(1 + p); 1 \\
\frac{1}{2}\beta, \frac{1}{2}(1 + \beta), 1 + p;
\end{array} \right] &= \frac{\Gamma(1 + \alpha + p)}{\Gamma(1 + \alpha)\Gamma(1 + p)},
\end{align*}
\]

valid for \( R(\beta - \alpha - p) > 0, R(\beta - \alpha) > 0 \) and \( R(\beta - p) > 0 \).

If \( \alpha = -n \) (a positive integer) in (6), it reduces to a known result of Bailey [2].

In case \( \beta = 1 + \alpha + p \) in (6), we have

\[
_3F_2\left[ \begin{array}{c}
a, \frac{1}{2}p, \frac{1}{2}(1 + p); 1 \\
\frac{1}{2}(1 + \alpha + p), \frac{1}{2}(2 + \alpha + p);
\end{array} \right] = \frac{\Gamma(1 + \alpha + p)}{\Gamma(1 + \alpha)\Gamma(1 + p)},
\]

valid for \( R(1 + \alpha) > 0, R(1 + p) > 0 \) and \( R(1 + \alpha + p) > 0 \).

In case \( \beta = 1 + p \) in (6), we have

\[
_3F_2\left[ \begin{array}{c}
\alpha, 1 + p - \alpha, \frac{1}{2}(1 + p); 1 \\
1 + p, \frac{1}{2}(2 + p);
\end{array} \right] = \frac{\Gamma(1 + p)\Gamma(1 - \alpha)}{\Gamma(1 + p - \alpha)},
\]

valid for \( R(1 + p) > 0, R(1 - \alpha) > 0 \) and \( R(1 + p - \alpha) > 0 \).

REFERENCES

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