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AN APPROXIMATION THEOREM OF RUNGE TYPE FOR THE HEAT EQUATION

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ABSTRACT. If Ω is an open subset of $\mathbf{R}^{n+1}$, the approximation problem is to decide whether every solution of the heat equation on Ω can be approximated by solutions defined on all of $\mathbf{R}^{n+1}$. The necessary and sufficient condition on Ω which insures this type of approximation is that every section of Ω taken by hyperplanes orthogonal to the t-axis be an open set without "holes," i.e., whose complement has no compact component. Part of the proof involves the Tychonoff counterexample for the initial value problem.

1. Introduction. A classical theorem of Runge asserts that if Ω is an open subset of the complex plane, then a necessary and sufficient condition for Ω to have the property that for any holomorphic function $f(z)$ on Ω there exists a sequence of polynomials $P_j(z)$ which converges to $f(z)$ locally uniformly on Ω is that Ω be simply connected. Or equivalently, we could require that the complement of Ω have no compact component. See [5, pp. 255–264], or [3, pp. 6–9].

Malgrange [4] extended this theorem to a large class of elliptic operators. A particular case of this states that if Ω is an open subset of $\mathbf{R}^n$, then a necessary and sufficient condition for Ω to have the property that for any harmonic function u on Ω there exists a sequence of harmonic functions u_j on $\mathbf{R}^n$ which converges to u locally uniformly on Ω is that the complement of Ω have no compact component.

The purpose of this paper is to present the corresponding theorem for the heat operator on $\mathbf{R}^{n+1}$,

$$H = \Delta - \frac{\partial}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}.$$

In doing this we shall require the counterexample of Tychonoff to uniqueness for the initial value problem for H. For $n = 1$ the example has the form

$$u(x, t) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{x^{2k}}{(2k)!},$$

where f is an appropriate infinitely differentiable function vanishing for $t \leq 0$ but not identically zero. E.g.,

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$$f(t) = \begin{cases} e^{-t^{-2}}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

See [1, pp. 50–51]. For our purposes it is necessary to require also that f vanish for $t \geq 1$, which we can do. And then for arbitrary n the required function can be obtained in the form $u(x_1, t)$. By a translation in t and rescaling we state this as a

Lemma. *For any $\delta > 0$ there exists a function u on $\mathbf{R}^{n+1}$ satisfying*
(i) $Hu = 0$ *on* $\mathbf{R}^{n+1}$,
(ii) $u(x, t) = 0$ *for* $|t| \geq \delta$,
(iii) $u(0, 0) \neq 0$.

2. **Topological preliminaries.** We shall be working with an open set $\Omega \subset \mathbf{R}^{n+1}$. For each $t \in \mathbf{R}$ let $\Omega(t)$ be the corresponding section of Ω: $\Omega(t) = \{x \in \mathbf{R}^n: (x, t) \in \Omega\}$. Thus, $\Omega(t)$ is an open subset of $\mathbf{R}^n$. If K is a compact set in $\mathbf{R}^{n+1}$, then the section $K(t)$ is a compact subset of $\mathbf{R}^n$.

Definition. Let $C \subset D$, where C is a compact set in $\mathbf{R}^n$ and D is an open set in $\mathbf{R}^n$. Then we denote by $\widehat{C_D}$ the union of C and all components of $D-C$ which are relatively compact in D.

This hull $\widehat{C_D}$ is of prime importance in the Runge theorem for harmonic functions in $\mathbf{R}^n$. It is always a compact set. In case D has the further property that $\mathbf{R}^n - D$ has no compact component, then $\widehat{C_D} = \widehat{C_{\mathbf{R}^n}}$. In this case we drop the subscript and simply write $\hat{C}$ for the union of C and all bounded components of $\mathbf{R}^n - C$.

Definition. If K is a compact set in $\mathbf{R}^{n+1}$, let

$$\hat{K} = \{(x, t): x \in \widehat{K(t)}\}.$$

Lemma. K *is compact.*

Proof. Each section $\widehat{K(t)}$ is contained in the convex hull of the section $K(t)$. Also, $K(t) = \emptyset$ for sufficiently large $|t|$. Therefore, $\hat{K}$ is bounded. To prove that $\hat{K}$ is closed, suppose that $(x, t) \notin \hat{K}$. This means $x \notin \widehat{K(t)}$, and this implies that there exists a continuous curve γ in $\mathbf{R}^n$ which starts at x and tends to ∞, lying in the open set $\mathbf{R}^n - K(t)$. Choose a closed ball $B \subset \mathbf{R}^n - K(t)$ centered at x. Then there exists $\epsilon > 0$ such that for $|t - t'| < \epsilon$ the set $K(t')$ is disjoint from B and the image of γ. But then for $x' \in B$ the point $x' \notin \widehat{K(t')}$. I.e., $B \times (t - \epsilon, t + \epsilon)$ is disjoint from $\hat{K}$. This proves $\mathbf{R}^{n+1} - \hat{K}$ is open. Q.E.D.

3. **The approximation theorem.**

Theorem. *Let $\Omega \subset \mathbf{R}^{n+1}$ be open. A necessary and sufficient condition that for any solution u of $Hu = 0$ on Ω there exists a sequence u_j of solutions of $Hu_j = 0$ on $\mathbf{R}^{n+1}$ which converges to u locally uniformly on Ω is* *that for every $t \in \mathbf{R}$, the complement of the section $\Omega(t)$ have no compact component.*

Proof. *Sufficiency.* We assume that each $\Omega(t)$ satisfies the condition. By Corollary 3.4.1 of [2] it suffices to consider any compactly supported distribution μ on $\mathbf{R}^{n+1}$ for which the support of $H^t\mu$ is contained in Ω, and to prove that the support of μ is also contained in Ω. Here H^t is the adjoint operator $\Delta + \partial/\partial t$. Let K be the support of $H^t\mu$. Since each $\Omega(t)$ has a complement free of compact components, $\widehat{K(t)} \subset \Omega(t)$. Thus, $\hat{K} \subset \Omega$ and $\hat{K}$ is compact. Now $H^t\mu = 0$ on $\mathbf{R}^{n+1} - K$. Since H^t is hypoelliptic, μ is an infinitely differentiable function on $\mathbf{R}^{n+1} - K$. Then for each t the function $\mu(x, t)$ is an analytic function of x for $x \in \mathbf{R}^n - K(t)$ (for solutions of the heat equation are analytic in the space variables) which vanishes for sufficiently large x (because μ has compact support). But unique continuation then requires $\mu(x, t) = 0$ on the domain $\mathbf{R}^n - \widehat{K(t)}$. Thus, $\mu = 0$ on $\mathbf{R}^{n+1} - \hat{K}$. Thus, the support of μ is contained in the compact set $\hat{K}$.

Necessity. We assume that Ω does not satisfy the required condition. Therefore, there exists some t such that $\mathbf{R}^n - \Omega(t)$ has a compact component A. By translation we can assume $0 \in A$ and $t = 0$, and we can write $\mathbf{R}^n - \Omega(0) = A \cup B$, where A and B are disjoint and B is closed. Choose an open set G with compact closure $\bar{G}$ such that $A \subset G \subset \bar{G} \subset \mathbf{R}^n - B$. Then the boundary of G is contained in $\Omega(0)$. Thus, $\bar{G} \subset \Omega(0) \cup G$. Thus, there exists an infinitely differentiable ϕ on $\mathbf{R}^n$ having support C contained in $\Omega(0) \cup G$, and satisfying $\phi \equiv 1$ on G. But then the support of the gradient $\nabla\phi$ is contained in $C - G \subset \Omega(0)$. Now there exists $\delta > 0$ such that $\operatorname{supp} \nabla\phi \times [-\delta, \delta] \subset \Omega$. By the lemma of §1 there exists a function u on $\mathbf{R}^{n+1}$ such that $H^t u \equiv 0$, $u(x, t) = 0$ for $|t| \geq \delta$, $u(0, 0) = 1$. Let $v(x, t) = \phi(x)u(x, t)$. Then

$$H^t v = 2\nabla\phi \cdot \nabla u + u\,\Delta\phi,$$

so that the support of $H^t v$ is contained in $\operatorname{supp} \nabla\phi \times [-\delta, \delta]$, which is contained in Ω. v itself has compact support and $v(0, 0) = 1$.

But then $H^t v$ is a smooth function and thus defines a Radon measure compactly supported in Ω, and as such it annihilates all solutions of the heat equation on $\mathbf{R}^{n+1}$ but not all solutions on Ω. For, if $Hw = 0$ on $\mathbf{R}^{n+1}$, then since v has compact support, $\langle H^t v, w\rangle = \langle v, Hw\rangle = \langle v, 0\rangle = 0$. But if E is a fundamental solution for H, then $HE = \delta$, the Dirac meansure at the origin, and so $HE = 0$ on Ω. But

$$\langle H^t v, E\rangle = \langle v, HE\rangle = \langle v, \delta\rangle = v(0, 0) = 1.$$

Therefore we conclude that the solutions on $\mathbf{R}^{n+1}$ are not dense in the solutions on Ω. Q.E.D.

4. **Remarks.** We have dealt with uniform convergence on compact sets in Ω. We could just as well have used the topology of $C^\infty(\Omega)$, involving uniform convergence of all derivatives on compact subsets of Ω. These topologies are the same for solutions of the heat equation.

Another result in Malgrange's thesis [4] is that in case the necessary and sufficient condition on Ω holds, then actually the exponential-polynomial solutions are dense in the solutions on Ω. In fact, one can even use just polynomial solutions since the polynomial $-\xi_1^2 - \cdots - \xi_n^2 - it$ is irreducible and vanishes at the origin. Also one can use just exponential solutions since this polynomial is irreducible; cf. [2, p. 78].

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