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REPRESENTATION OF A CRINKLED ARC

RICHARD A. VITALE

ABSTRACT. Johnson [*A crinkled arc*, Proc. Amer. Math. Soc. 25 (1970), 375–376] has shown that under suitable normalizations all crinkled arcs are unitarily equivalent. Using this result, we find a general series expansion for a crinkled arc:

$$f(t) = \sqrt{2} \sum_{n=1}^{\infty} x_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi},$$

where $\{x_n\}$ is an orthonormal set.

Originally introduced in problem four of Halmos [2], a crinkled arc may be defined as a continuous map $f\colon [0, 1] \to X$, a Hilbert space, which is one-to-one and possesses the *crinkly* property: if $0 \leq a < b \leq c < d \leq 1$, then the chords $f(b) - f(a)$ and $f(d) - f(c)$ are orthogonal. It is convenient to consider the following normalizations:

(I) $f(0) = 0$ by translation,

(II) $\|f(1)\| = 1$ by a scale change,

(III) $X = \bigvee f$ where $\bigvee f$ is the smallest Hilbert space containing the values of f.

Under these conditions, Johnson [3] has derived a number of results including $t \to \|f(t)\|$ is a strictly monotone continuous map of $[0, 1]$ onto $[0, 1]$. This allows an additional normalization in the following way: if $f(t)$ is a crinkled arc with $\phi(t) = \|f(t)\|$, then $\hat{f}(t) = f(\phi^{-1}(t^{1/2}))$ represents the same locus but with $\|\hat{f}(t)\|^2 = t$. Consequently, we introduce

(IV) $\|f(t)\|^2 = t$

and consider now only crinkled arcs satisfying (I)–(IV). In this context, Johnson's main result says that any two crinkled arcs are unitarily equivalent in the sense that if $f\colon [0, 1] \to X$, $g\colon [0, 1] \to Y$, then there is an isometry $U\colon X \overset{\text{onto}}{\to} Y$ such that $g(t) = Uf(t)$. We shall use this result to prove the following representation.

Theorem. *$f(t)$ is a crinkled arc iff*

$$f(t) = \sqrt{2} \sum_{n=1}^{\infty} x_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi} \tag{1}$$

where $\{x_n\} \subseteq X$ is an orthonormal set.

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The proof follows from the observation that in the theory of stochastic processes, Brownian motion $W(t)$ defined on $[0, 1]$ may be regarded as a crinkled arc in a suitable Hilbert space B of random variables. A direct application of the Karhunen-Loève expansion theorem provides the representation

$$W(t) = \sqrt{2} \sum_{n=1}^{\infty} b_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi}$$

where $\{b_n\} \subseteq B$ is an orthonormal set (Ash [1]). If $f(t) \subseteq X$ is a crinkled arc, there is an isometry $U: B \to X$ such that

$$f(t) = UW(t) = \sqrt{2} \sum_{n=1}^{\infty} (Ub_n) \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi} .$$

Identifying $x_n = Ub_n$, we have the desired result in one direction. Conversely, if $f(t)$ has a representation (1) then we can define an isometry $U: B \xrightarrow{\text{onto}} X$ coordinatewise by $Ub_n = x_n$. It is immediate that all of the properties of $W(t)$ as a crinkled arc are carried into $f(t)$.

Remark 1. The series convergence in (1) is uniform in t since

$$\left\| \sqrt{2} \sum_{n=k}^{\infty} x_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi} \right\|^2 = 2 \sum_{n=k}^{\infty} \|x_n\|^2 \frac{\sin^2(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})^2\pi^2}$$

$$\leq 2 \sum_{n=k}^{\infty} \frac{1}{(n - \frac{1}{2})^2\pi^2} \to 0 \quad \text{as } k \to \infty.$$

Remark 2. If (IV) is dropped, then (1) holds with t replaced by $\|f(t)\|^2$ on the right-hand side. Relaxations of (I) and (II) require the obvious modifications.

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DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912