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# CHARACTERIZATIONS OF C-COMPACT SPACES

LARRY L. HERRINGTON AND PAUL E. LONG

ABSTRACT. This paper gives characterizations of $C$-compact spaces, some of which make use of nets. Sufficient conditions for a space to be $C$-compact are given which make use of the graphs of weakly-continuous functions and a class of spaces which contains the class of Hausdorff completely normal and fully normal spaces.

1. **Introduction.** Our primary interest is the investigation of $C$-compact spaces. We give several characterizations of such spaces, some of which make use of nets and a type of convergence we define as $r$-convergence. Some of these characterizations lead immediately to characterizations of $H$-closed and minimal Hausdorff spaces. We also obtain sufficient conditions for a space to be $C$-compact through the graph of a weakly-continuous function and a class of spaces which includes the class of Hausdorff completely normal and fully normal spaces.

2. **Preliminary definitions and theorems.**

**Definition 1** [8]. A space $X$ is *C-compact* if for each closed subset $A \subset X$ and each open cover $\{U_\alpha | \alpha \in \Delta\}$ of $A$, there exists a finite subcollection $\{U_{\alpha_i} | i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$.

**Definition 2** [1, p. 97]. A $T_2$-space $X$ is *H-closed* if for each open cover $\{U_\alpha | \alpha \in \Delta\}$ of $X$, there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $X = \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$.

Recall that a set $U$ is regular-open if $\mathrm{Int}(\mathrm{cl}(U)) = U$. In view of the fact that for any open set $U$, $\mathrm{Int}(\mathrm{cl}(U))$ is regular-open [3, p. 92], it follows immediately that the open sets in Definition 2 may be replaced with regular-open sets and an equivalent definition obtained. We now show the same may be done with Definition 1.

**Lemma 1.** *The space $X$ is C-compact if and only if for each closed $A \subset X$ and regular-open cover $\{U_\alpha | \alpha \in \Delta\}$, there exists a finite subcollection $\{U_{\alpha_i} | i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$.*

**Proof.** If $X$ is $C$-compact, the condition follows from Definition 1.

Now suppose the condition holds and let $\{U_\alpha | \alpha \in \Delta\}$ be any open cover

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of $A$. Then $\{\mathrm{Int}(\mathrm{cl}(U_\alpha))|\alpha \in \Delta\}$ is a regular-open cover of $A$ so there exists a finite subcollection $\{\mathrm{Int}(\mathrm{cl}(U_{\alpha_i}))|i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^{n} \mathrm{cl}[\mathrm{Int}(\mathrm{cl}(U_{\alpha_i}))]$. But for each $i$, $\mathrm{cl}[\mathrm{Int}(\mathrm{cl}(U_{\alpha_i}))] = \mathrm{cl}(U_{\alpha_i})$. Therefore, $A \subset \bigcup_{i=1}^{n} \mathrm{cl}(U_{\alpha_i})$ which shows $X$ is $C$-compact.

**Definition 3.** Let $X$ be a space and let $\emptyset \neq A \subset X$. Let $\mathfrak{F} = \{A_\alpha \subset A|\ \alpha \in \Delta\}$ be a filterbase in $A$. Then $\mathfrak{F}$ *r-converges* to $a \in A$ ($\mathfrak{F} \to_r a$) if for each open $V$ in $X$ containing $a$ there exists an $A_\alpha \in \mathfrak{F}$ such that $A_\alpha \subset \mathrm{cl}(V)$. The filterbase $\mathfrak{F}$ *r-accumulates* to $a \in A$ ($\mathfrak{F} \propto_r a$) if for each open $V$ in $X$ containing $a$ and for each $A_\alpha \in \mathfrak{F}$, $A_\alpha \cap \mathrm{cl}(V) \neq \emptyset$.

The convergence (accumulation) of filterbases in the usual sense, of course, implies $r$-convergence ($r$-accumulation). The converse does not hold, however, as the following example shows.

**Example 1.** Let $N$ denote the natural numbers and let $R$ denote the reals with the cocountable topology. Define a filterbase $\mathfrak{F}$ in $A = [0, \infty)$ by $\mathfrak{F} = \{A_n|A_n = \{n, n+1, n+2, \cdots\}\}$, $n \in N$. Then for any $a \in A$, $\mathfrak{F} \not\to a$ and $\mathfrak{F} \not\propto a$. However, it is easily shown that $\mathfrak{F} \to_r a$ and $\mathfrak{F} \propto_r a$.

We now list several results concerning $r$-convergence and $r$-accumulation whose straightforward proofs are omitted.

**Theorem 1.** *Let $X$ be a topological space and let $A \subset X$. Then*

(a) *If $\mathfrak{F}$ is a filterbase in $A$ such that $\mathfrak{F} \to_r a \in A$, then $\mathfrak{F} \propto_r a$.*

(b) *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filterbases in $A$ and suppose $\mathfrak{F}_2$ is stronger than $\mathfrak{F}_1$ ($\mathfrak{F}_2 \prec \mathfrak{F}_1$). If $\mathfrak{F}_2 \propto_r a \in A$, then $\mathfrak{F}_1 \propto_r a$.*

(c) *Let $\mathfrak{M}$ be a maximal filterbase in $A$. Then $\mathfrak{M} \propto_r a \in A$ if and only if $\mathfrak{M} \to_r a$.*

**Definition 4.** If $D$ is a directed set and $\phi: D \to A \subset X$ is a net, then

(a) $\phi$ *r-converges* to $a \in A$ ($\phi \to_r a$) if for each open $V \subset X$ containing $a$, there exists a $b \in D$ such that $\phi(T_b) \subset \mathrm{cl}(V)$ where $T_b = \{c \in D|c \geq b\}$.

(b) $\phi$ *r-accumulates* to $a \in A$ ($\phi \propto_r a$) if for each open $V \subset X$ containing $a$ and for every $b \in D$, $\phi(T_b) \cap \mathrm{cl}(V) \neq \emptyset$.

Of course, if $\phi: D \to A$ is a net in $A \subset X$, the family $\mathfrak{F}(\phi) = \{\phi(T_b)|\ b \in D\}$ is a filterbase in $A$ and it is routine to verify that

(1) $\mathfrak{F}(\phi) \to_r a \in A$ if and only if $\phi \to_r a$, and

(2) $\mathfrak{F}(\phi) \propto_r a \in A$ if and only if $\phi \propto_r a$.

Conversely, every filterbase $\mathfrak{F}$ in $A \subset X$ determines a net $\phi: D \to A$ such that

(1) $\mathfrak{F} \to_r a \in A$ if and only if $\phi \to_r a$, and

(2) $\mathfrak{F} \propto_r a \in A$ if and only if $\phi \propto_r a$.

The construction of such a net is the same as that of [3, p. 213].

3. **Filterbase and net characterizations of $C$-compact spaces.** Theorem 1 of [7] gives several characterizations of $C$-compact spaces. We offer the following characterizations:

**Theorem 2.** *In a space $X$ the following are equivalent:*

(a) *$X$ is $C$-compact.*

(b) *For each closed $A \subset X$ and each regular-open cover $\{U_\alpha | \alpha \in \Delta\}$ of $A$ there exists a finite subcollection $\{U_{\alpha_i} | i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$.*

(c) *For each closed $A \subset X$ and each collection of nonempty regular-closed sets $\{F_\alpha | \alpha \in \Delta|$ such that $(\bigcap_\alpha F_\alpha) \cap A = \emptyset$, there exists a finite subcollection $\{F_{\alpha_i} | i = 1, 2, \ldots, n\}$ such that $(\bigcap_{i=1}^n \mathrm{Int}(F_{\alpha_i})) \cap A = \emptyset$.*

(d) *For each closed $A \subset X$ and each collection of nonempty regular-closed sets $\{F_\alpha | \alpha \in \Delta\}$, if each finite subcollection $\{F_{\alpha_i} | i = 1, 2, \ldots, n\}$ has the property that $(\bigcap_{i=1}^n \mathrm{Int}(F_{\alpha_i})) \cap A \neq \emptyset$, then $(\bigcap_\alpha F_\alpha) \cap A \neq \emptyset$.*

(e) *For each closed $A \subset X$ and each filterbase $\mathcal{F} = \{A_\alpha | \alpha \in \Delta\}$ in $A$, there exists an $a \in A$ such that $\mathcal{F} \propto_r a$.*

(f) *For each closed $A \subset X$ and each maximal filterbase $\mathfrak{M} = \{A_\alpha | \alpha \in \Delta\}$ in $A$ there exists an $a \in A$ such that $\mathfrak{M} \to_r a$.*

**Proof.** (a) if and only if (b) has been shown in Lemma 1.

(a) implies (c) follows from Theorem 1 of [7].

(c) implies (b). Let $\{U_\alpha | \alpha \in \Delta\}$ be a regular-open cover of $A$. Then $A \subset \bigcup_\alpha U_\alpha$ implies $(\bigcap_\alpha (X - U_\alpha)) \cap A = \emptyset$. Since $X - U_\alpha$ is regular-closed for each $\alpha \in \Delta$, the hypothesis of (c) implies there is a finite subcollection $\{X - U_{\alpha_i} | i = 1, 2, \ldots, n\}$ such that $(\bigcap_{i=1}^n \mathrm{Int}(X - U_{\alpha_i})) \cap A = \emptyset$. It follows that $A \subset \bigcup_{i=1}^n (X - \mathrm{Int}(X - U_{\alpha_i}))$. However, $\mathrm{Int}(X - U_{\alpha_i}) = X - (\mathrm{cl}(X - (X - U_{\alpha_i}))) = X - \mathrm{cl}(U_{\alpha_i})$ for each $i = 1, 2, \ldots, n$. Therefore, $A \subset \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$ which is condition (b).

(c) if and only if (d) is clear.

(a) implies (e). Suppose there exists a filterbase $\mathcal{F} = \{A_\alpha | \alpha \in \Delta\}$ in $A$ such that $\mathcal{F} \not\propto_r a$ for all $a \in A$. Then for each $a \in A$ there exists an open set $U(a)$ and some $A_{\alpha(a)} \in \mathcal{F}$ such that $A_{\alpha(a)} \cap \mathrm{cl}(U(a)) = \emptyset$. The collection $\{U(a) | a \in A\}$ is an open cover of $A$, so by (a) there exists a finite subcollection $\{U(a_i) | i = 1, 2, \ldots, n\}$ such that $A \subset \bigcup_{i=1}^n \mathrm{cl}(U(a_i))$. Let $A_{\alpha_0} \in \mathcal{F}$ such that $A_{\alpha_0} \subset \bigcap_{i=1}^n A_{\alpha(a_i)}$. Since $A_{\alpha_0} \neq \emptyset$, there is some $1 \leq j \leq n$

such that $A_{\alpha_0} \cap \mathrm{cl}(U(a_j)) \neq \emptyset$. This implies $A_{\alpha(a_j)} \cap \mathrm{cl}(U(a_j)) \neq \emptyset$ which is a contradiction. Condition (e) follows.

(e) implies (d). Suppose there exist a closed set $A \subset X$ and a collection of regular-closed sets $\{F_\alpha | \alpha \in \Delta\}$ such that each finite subcollection $\{F_{\alpha_i} | i = 1, 2, \ldots, n\}$ has the property that $(\bigcap_{i=1}^n \mathrm{Int}(F_{\alpha_i})) \cap A \neq \emptyset$ but $(\bigcap_\alpha F_\alpha) \cap A = \emptyset$. Then the sets $(\mathrm{Int}(F_\alpha)) \cap A$, $\alpha \in \Delta$, together with all finite intersections of the form $(\bigcap_{i=1}^n \mathrm{Int}(F_{\alpha_i})) \cap A$, form a filterbase $\mathfrak{F}$ in $A$. By (e), this filterbase $r$-accumulates to some point $a \in A$. Thus, for each open $U(a)$ containing $a$ and each $\mathrm{Int}(F_\alpha)$, $\mathrm{cl}(U(a)) \cap (\mathrm{Int}(F_\alpha) \cap A) \neq \emptyset$. The fact that $F_\alpha \cap A \neq \emptyset$ for all $\alpha \in \Delta$ and the assumption that $(\bigcap_\alpha F_\alpha) \cap A = \emptyset$ give the existence of an $\alpha_0 \in \Delta$ such that $a \notin F_{\alpha_0}$. Therefore, $a \notin \mathrm{Int}(F_{\alpha_0})$ so that $a \in (X - F_{\alpha_0}) \subset (X - \mathrm{Int}(F_{\alpha_0}))$. It then follows that $a \in (X - F_{\alpha_0}) \subset \mathrm{cl}(X - F_{\alpha_0}) \subset (X - \mathrm{Int}(F_{\alpha_0}))$ which implies $\mathrm{cl}(X - F_{\alpha_0}) \cap \mathrm{Int}(F_{\alpha_0}) = \emptyset$. But this means $\mathfrak{F} \not\propto_r a$. The contradiction gives $(\bigcap_\alpha F_\alpha) \cap A \neq \emptyset$.

(e) implies (f). Let $\mathfrak{M} = \{A_\alpha | \alpha \in \Delta\}$ be a maximal filterbase in the closed set $A \subset X$. Then $\mathfrak{M} \propto_r a \in A$ by (e) so that $\mathfrak{M} \to_r a$ by Theorem 1(c).

(f) implies (e). Let $\mathfrak{F} = \{A_\alpha | \alpha \in \Delta\}$ be a filterbase in the closed set $A \subset X$. Then there exists a maximal filterbase $\mathfrak{M}$ such that $\mathfrak{M} \prec \mathfrak{F}$. By (f), $\mathfrak{M} \to_r a \in A$. Applying parts (a) and (b) of Theorem 1, we see that $\mathfrak{F} \propto_r a$.

Our knowledge of the $r$-convergence and $r$-accumulation of nets immediately gives the next theorem.

**Theorem 3.** *In a space $X$ the following are equivalent:*

(a) *$X$ is $C$-compact.*

(b) *For each closed $A \subset X$ and each net $\phi$ in $A$, there is a point $a \in A$ such that $\phi \propto_r a$.*

(c) *For each closed $A \subset X$ and each universal net $\phi$ in $A$, there exists a point $a \in A$ such that $\phi \to_r a$.*

If we replace the set $A$ in Theorem 2 with the entire space $X$, the immediate characterizations of $H$-closed spaces are obtained:

**Theorem 4.** *In a $T_2$-space $X$ the following are equivalent:*

(a) *$X$ is $H$-closed.*

(b) *Each filterbase $\mathfrak{F} = \{A_\alpha | \alpha \in \Delta\}$ in $X$ $r$-accumulates to some $x \in X$.*

(c) *Each maximal filterbase $\mathfrak{M}$ in $X$ $r$-converges.*

**Corollary.** *In a $T_2$-space $X$ the following are equivalent:*

(a) *$X$ is H-closed.*

(b) *Each net in $X$ has an r-accumulation point.*

(c) *Each universal net in $X$ r-converges.*

**Definition** 5 [1, p. 99]. A $T_2$-space $X$ is *minimal* $T_2$ if and only if it is $H$-closed and semiregular.

It follows from Exercise 19 of [2, p. 146] that a $T_2$-space $X$ is minimal $T_2$ if and only if each open filterbase in $X$ possessing at most one accumulation point is convergent. Thus, characterizations of minimal $T_2$-spaces may be given in terms of open filterbases. Open filterbases, of course, determine nets but not every net determines an open filterbase. In terms of nets and $r$-convergence, we give the following characterization of minimal $T_2$-spaces.

**Theorem** 5. *Let $X$ be a $T_2$-space. Then $X$ is minimal $T_2$ if and only if each net in $X$ possessing at most one r-accumulation point is convergent.*

**Proof.** Suppose the condition is given and let $\mathcal{F} = \{O_\alpha : \alpha \in \Delta\}$ be an open filterbase in $X$ possessing at most one accumulation point. Since an open filterbase accumulates if and only if it $r$-accumulates, $\mathcal{F}$ possesses at most one $r$-accumulation point. Consequently, the filterbase $\mathcal{F}$ determines a net $\phi: D \to X$ with the property that $\phi$ possesses at most one $r$-accumulation point. Therefore, by hypothesis, $\phi$ converges to some point $a$ in $X$. Now if $V(a)$ is an open set containing $a$, there exists some $(c, O_\alpha) \in D$ such that $O_\alpha = \phi(T_{(c,O_\alpha)}) \subset V(a)$. But this implies that $\mathcal{F} \to a$ showing that $X$ is minimal $T_2$.

Conversely, assume that $X$ is minimal $T_2$ and suppose that the net $\phi: D \to X$ has at most one $r$-accumulation point. Since $X$ is $H$-closed, $\phi$ $r$-accumulates to some point $a \in X$. We need to show $\phi$ converges to $a$ in the usual sense. To do this, we use the fact that our space is semiregular. Thus, suppose there exists a regular-open set $U(a)$ containing $a$ such that for every $d \in D$, $\phi(T_d) \not\subset U(a)$. Then there is some subnet $\phi_s$ of the net $\phi$ and a directed set $D_s \subset D$ such that $\phi_s: D_s \to (X - U(a))$ is a net in $X - U(a)$. Since $X - U(a)$ is regular-closed, it follows from Theorem 3.3(d) of [1, p. 97] that $X - U(a)$ is $H$-closed. Consequently, by the Corollary to Theorem 4, the net $\phi_s$ $r$-accumulates to some point $x \in X - U(a)$. Hence, $\phi \propto_r x$ which is a contradiction since $\phi$ has the unique $r$-accumulation point $a \neq x$. We conclude $\phi \to a$.

**Corollary.** *Let $X$ be a $T_2$-space. Then $X$ is minimal $T_2$ if and only if each filterbase in $X$ possessing at most one r-accumulation point is convergent.*

**4. Weakly continuous functions and strongly closed graphs.**

**Definition 6** [6, p. 44]. A function $f\colon X \to Y$ is *weakly-continuous* if for each $x \in X$ and each open $V$ containing $f(x)$, there exists an open $U$ containing $x$ such that $f(U) \subset \mathrm{cl}(V)$.

We have immediately that $f\colon X \to Y$ is weakly-continuous at $x \in X$ if and only if for each net $\{x_\alpha\}$ in $X$ such that $\{x_\alpha\} \to x$, the net $\{f(x_\alpha)\} \to_r f(x)$.

**Definition 7.** A function $f\colon X \to Y$ has a strongly-closed graph if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing $x$ and $y$, respectively, such that $(U \times \mathrm{cl}(V)) \cap G(f) = \emptyset$.

**Example 2.** Let $X = [0, 1]$ have the usual subspace topology and let $Y = [0, 1]$ have the topology generated by the usual open sets together with the set $A = \{r \mid r$ is rational and $1/3 < r < 2/3\}$ as a subbase. The identity function $i\colon X \to Y$ has a strongly-closed graph. Furthermore $i$ is not continuous and $Y$ is not $C$-compact, minimal $T_2$ nor semiregular.

We remark that for any $T_2$-space $X$, the identity may $i\colon X \to X$ has a strongly-closed graph.

**Theorem 6.** *Let $f\colon X \to Y$ be a function and let $Y$ be $C$-compact. If $f$ has a strongly-closed graph, then $f$ is continuous.*

**Proof.** Suppose there exists an open $V \subset Y$ such that $f^{-1}(V)$ is not open. Then there exists an $x \in f^{-1}(V)$ such that $x \in \mathrm{cl}(X - f^{-1}(V))$. Thus there exists a net $\{x_\alpha\}$ in $X - f^{-1}(V)$ such that $x_\alpha \to x$. Now $\{f(x_\alpha)\}$ is a net in $Y - V$ which implies there exists a point $z \in Y - V$ such that $\{f(x_\alpha)\} \propto_r z$ by Theorem 3(b). Since $(x, z) \notin G(f)$, there exist open sets $U(x)$ and $V(z)$ such that $(U(x) \times \mathrm{cl}(V(z))) \cap G(f) = \emptyset$ because $f$ has a strongly-closed graph. However, $\{x_\alpha\} \to x$ implies there exists an $\alpha_0$ such that for every $\alpha \geq \alpha_0$, $x_\alpha \in U(x)$, and $\{f(x_\alpha)\} \propto_r z$ implies there exists some $\alpha_1 \geq \alpha_0$ such that $f(x_{\alpha_1}) \in \mathrm{cl}(V(z))$, from which it follows that $(x_{\alpha_1}, f(x_{\alpha_1})) \in U(x) \times \mathrm{cl}(V(z))$. This contradicts our earlier conclusion that $(U(x) \times \mathrm{cl}(V(z))) \cap G(f) = \emptyset$. Hence, $f^{-1}(V)$ is open.

**Theorem 7.** *Let $f\colon X \to Y$ be a function and let $Y$ be minimal $T_2$. If $f$ has a strongly-closed graph, then $f$ is continuous.*

**Proof.** If we recall Definition 5 and part (b) of the Corollary to Theorem 4, the proof then parallels that of Theorem 6 if we take the open set $V \subset Y$ to be regular-open.

Our next example shows the strongly-closed graph condition in Theorem 6 cannot be relaxed to a closed graph condition. Also, since a $C$-compact $T_2$-space is minimal $T_2$ [8], the same example will show that the strongly-closed graph condition in Theorem 7 cannot be relaxed to a closed graph condition.

**Example** 3. Let $X$ be the space of Example 1 of [7] and let $A = \{0\} \cup \{1/n | n \in N\}$ have the subspace topology of the reals. We note that $A$ is compact and hence fully normal. Define $f: A \to X$ by $f(1/n) = (1/n, 0)$ and and $f(0) = (1, 1)$. Then $f$ is closed with closed point inverses and $A$ is regular so that $f$ has a closed graph by Corollary 3.9 of [4]. However, $f$ is not weakly-continuous at $0 \in A$ and thus not continuous. Since $G(f)$ is closed there exist open sets $U \subset A$ and $V \subset X$ containing $0 \in A$ and $(0, 0) \in X$, respectively, such that $(U \times V) \cap G(f) = \emptyset$. However, it follows from the nature of the topology on $X$ that $(U \times \mathrm{cl}(V)) \cap G(f) \neq \emptyset$. Therefore, $G(f)$ is not strongly closed. This example also shows that the class of functions with strongly-closed graphs is smaller than the class of functions with closed graphs.

We now show how to use weakly-continuous functions and functions with strongly-closed graphs to obtain sufficient conditions for a space to be $C$-compact. To do this we rely on a particular class of spaces described by Professor Kasahara in [5]. This class of spaces, denoted by $\mathcal{S}$, contains the class of all Hausdorff completely normal and fully normal spaces.

**Theorem 8.** *A $T_1$-space $Y$ is C-compact if for every $X \in \mathcal{S}$, each function $f: X \to Y$ with a closed graph is weakly-continuous.*

**Proof.** Suppose $Y$ is not $C$-compact. Then by Theorem 3(b) there exist a closed set $A \subset Y$ and some net $f: D \to A \subset Y$ which has no $r$-accumulation point. That is, for each $a \in A$ there exist an open set $U(a) \subset Y$ and some $b \in D$ such that $f(T_b) \cap \mathrm{cl}(U(a)) = \emptyset$. Let $\infty \notin D$ and let $X = D \cup \{\infty\}$. It follows that the power set of $D$, $P(D)$, together with $\{T_d \cup \{\infty\} | d \in D\}$ is a base for a fully normal and completely normal $T_2$-space. Let $p \in A$ and define $g: X \to A$ by $g|D = f$ and $g(\infty) = p$. Let $(x, a) \notin G(g)$ where $a \in A$ and consider the two cases $x \in D$ and $x = \infty$. If $x \in D$, the fact that $g(x) \neq a$ gives the existence of an open set $V(a) \subset Y - \{g(x)\}$. Now $\{x\}$ is open in $X$, which implies $(\{x\} \times V(a)) \cap G(g) = \emptyset$. If $x = \infty$, $g(\infty) = p \neq a$. Since $a \in A$ is not an $r$-accumulation point of the net $f$, there exist an open $U(a) \subset Y - \{p\}$ and some $d \in D$ so that $f(T_d) \cap \mathrm{cl}(U(a)) = \emptyset$. The fact that $g|D = f$ implies $g(T_d) \cap \mathrm{cl}(U(a)) = \emptyset$ which furthermore implies $((T_d \cup \{\infty\}) \times U(a)) \cap G(g) = \emptyset$. We have now shown $G(g)$ is closed in the closed subset $X \times A$ of $X \times Y$. Therefore, $G(g)$ is closed in $X \times Y$.

Now the identity function $i: D \to D \subset X$ defines a net in $X$ and it is clear that $i \to \infty \in X$. Thus, given an open set $H(\infty)$, there exists some $d \in D$ such that $i(T_d) = T_d \subset H(\infty)$. Since $p$ is not an $r$-accumulation point of the net $f$, there exist an open set $W(p)$ and some $d_0 \in D$ such that for all

$d \geq d_0$, $f(T_d) \cap \mathrm{cl}(W(p)) = \emptyset$. But $f(T_d) = g(T_d)$ so that $g(T_d) \cap \mathrm{cl}(W(p)) = \emptyset$. Consequently $g$ is not weakly-continuous at $x = \infty$. We have now shown that $g: X \to A \subset Y$, $X \in \mathcal{S}$, $G(g)$ is closed, but $g$ is not weakly-continuous. This is contrary to hypothesis and shows our assumption that $Y$ is not $C$-compact is false.

**Remark 1.** Let $f: X \to A \subset Y$ be a function from $X$ into a subset $A$ of $Y$. We say that $G(f)$ is *strongly-closed with respect to* $A$ if for each point $(x, a) \notin G(f)$ (where $a \in A$), there exist open sets $U \subset X$ and $V \subset Y$ containing $x$ and $a$, respectively, such that $(U \times \mathrm{cl}(V)) \cap G(f) = \emptyset$. Of course, if $G(f)$ is strongly-closed with respect to $A$ and if $A$ is closed in $Y$, then $G(f)$ is closed in $X \times Y$. However, if $G(f)$ is strongly-closed with respect to $A$, then $G(f)$ need not be strongly-closed with respect to $Y$. To see this, let $I^2 = [0, 1] \times [0, 1]$ have as a subbase the usual open sets in $I^2$ along with the collection of sets $\{I^2 - B \times \{0\}: B \subset [0, 1]\}$. It follows that $I^2$ is an $H$-closed Urysohn space with the property that for each $B \subset [0, 1]$, $\mathrm{cl}(I^2 - B \times \{0\}) = I^2$. Now let $I = [0, 1]$ have the usual topology and let $A = \{(x, 0) \in I^2: x \neq 0\} \cup \{(1, 1)\} \subset I^2$. (We note that $A$ is closed in $I^2$.) Define $f: I \to A \subset I^2$ by $f(x) = (x, 0)$ if $x \neq 0$ and $f(0) = (1, 1)$. By Corollary 3.9 of [7], $G(f)$ is closed in $I \times I^2$. Moreover, it is easy to see that $G(f)$ is strongly-closed with respect to $A$. However, if $U$ and $V$ are open sets containing 0 and (0, 0), respectively, such that $(U \times V) \cap G(f) = \emptyset$, then $(U \times \mathrm{cl}(V)) \cap G(f) \neq \emptyset$. This shows that $G(f)$ is not strongly-closed in $I \times I^2$.

We say $f: X \to A \subset Y$ is weakly-continuous with respect to $A$ (with respect to $Y$) if $f: X \to A$ (resp., $f: X \to Y$) is weakly-continuous. We note that the function in Remark 1 is weakly-continuous with respect to $I^2$ at $x = \frac{1}{2}$, but $f$ is not weakly-continuous with respect to $A$ at $x = \frac{1}{2}$. With these observations in mind, we give

**Theorem 9.** *A $T_2$-space $Y$ is $C$-compact if for every topological space $X$ belonging to class $\mathcal{S}$ and for every closed $A \subset Y$, each $f: X \to A \subset Y$ with a strongly-closed graph with respect to $A$ is weakly-continuous with respect to $Y$.*

**Proof.** Since $Y$ is $T_2$, we can choose the open set $V(a) \subset Y - \{g(x)\}$ in Theorem 8 such that $g(x) \notin \mathrm{cl}(V(a))$. Thus $(\{x\} \times \mathrm{cl}(V(a))) \cap G(g) = \emptyset$. We can also choose the open set $U(a) \subset Y - \{p\}$ in $Y$ such that $p \notin \mathrm{cl}(U(a))$. It then follows that $((T_d \cup \{\infty\}) \times \mathrm{cl}(U(a))) \cap G(g) = \emptyset$ which shows $G(g)$ is strongly-closed with respect to $A$.

If the space $Y$ in Theorem 1 of [5] is Urysohn, it is natural to ask if the sufficient part of the hypothesis can be weakened. We offer

**Theorem 10.** *A Urysohn space $Y$ is compact if for every $X$ in class $\mathcal{S}$*

*and for every closed $A \subset Y$, each $f: X \to A$ with a strongly-closed graph with respect to $A$ is weakly-continuous with respect to $Y$.*

**Proof.** Since a Urysohn space $Y$ is $C$-compact if and only if it is compact [8], it is enough to show that $Y$ is $C$-compact. Thus, our theorem follows by applying Theorem 9.

We note that the function defined in Remark 1 shows that not every function into a Urysohn space $Y$ with a closed graph has a strongly-closed graph.

We conclude by noting that the function in Example 3 is closed and has point inverses but is not an open function. With this in mind, we have

**Theorem 11.** *Let $f: X \to Y$ be an open and closed function from a regular space $X$ into a $C$-compact space $Y$. If $f$ has closed point inverses, then $f$ is continuous.*

**Proof.** Suppose there is an $x \in X$ and some $U \subset Y$ containing $f(x)$ such that for every open $V$ containing $x$, $f(V) \cap (Y - U) \neq \emptyset$. Let $I = \{f(\mathrm{cl}(V)) | V$ is open and contains $x\}$. Since $f$ is closed, $I$ is a collection of closed sets. Now $f(V) \subset f(\mathrm{cl}(V))$ and $f$ open implies $f(x) \in f(V) \subset \mathrm{Int}(f(\mathrm{cl}(V))) \subset f(\mathrm{cl}(V))$ for every open $V$ containing $x$. This leads us to conclude that for every finite collection of open sets $\{V_i | i = 1, 2, \ldots, n\}$ each containing $x$, $\bigcap_{i=1}^n \mathrm{Int}(f(\mathrm{cl}(V_i))) \cap (Y - U) \neq \emptyset$. The reason is that if this were not true, then $\bigcap_{i=1}^n V_i$ is open, contains $x$ and

$$\emptyset \subset (Y - U) \cap f\left(\bigcap_{i=1}^{n} (V_i)\right) \subset (Y - U) \cap \left(\bigcap_{i=1}^{n} f(V_i)\right)$$

$$\subset (Y - U) \cap \left(\bigcap_{i=1}^{n} f(\mathrm{Int}(\mathrm{cl}(V_i)))\right) \subset (Y - U) \cap \left(\bigcap_{i=1}^{n} \mathrm{Int}(f(\mathrm{cl}(V_i)))\right) = \emptyset$$

which shows there exists an open set $W = \bigcap_{i=1}^n V_i$ containing $x$ such that $f(W) \cap (Y - U) = \emptyset$ contrary to our initial assumption. Now, since $Y$ is $C$-compact [7, Theorem 1], $A = \bigcap \{f(\mathrm{cl}(V)) \cap (Y - U) | V$ is open and contains $x\} \neq \emptyset$.

Consider any $y \in A$. Then $y \notin U$ so that $f(x) \neq y$ and hence $\{x\} \cap f^{-1}(y) = \emptyset$. Since $X$ is regular, there exist open sets $G$ and $W$ containing $x$ and $f^{-1}(y)$, respectively, such that $G \cap W = \emptyset$, hence $\mathrm{cl}(G) \cap W = \emptyset$. Consequently, $y \notin f(\mathrm{cl}(G))$ which implies that $y \notin A$. The contradiction reached concerning $A$ means our initial assumption is false so that $f$ is continuous.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701 (Current address of P. E. Long)

*Current address* (L. L. Herrington): Department of Mathematics, University of Arkansas, Pine Bluff, Arkansas 71601