PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

WHEN IS THE MAXIMAL RING OF QUOTIENTS PROJECTIVE?

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ABSTRACT. Let R be an associative ring with 1, and Q its maximal ring of right quotients. If r belongs to R, a right insulator for r in R is a finite subset of R, $\{r_i\}_{i=1}^m$, such that the right annihilator of $\{rr_i; i = 1, \ldots, m\}$ is zero. Then we have: If Q is a projective right R-module, Q is finitely generated; if R is nonsingular, then Q is projective as a right R-module if and only if there exists $e = e^2$ in R such that eR is injective and e has a right insulator in R; under these circumstances, $R = Q$ if and only if e has a left insulator in R. We prove some related results for torsionless Q, and give an example of a prime ring R such that Q is a cyclic projective right R-module, but $R \neq Q$.

Let R be an associative ring with 1; Q (or $Q(R)$) will denote the complete (or maximal) ring of right quotients of R (e.g. [6, p. 94 on]). For a subset A of R, we denote its right (left) annihilator by A^r (A^l). Definitions of the terms essential, dense, singular submodule, etc. can be found in [6].

The term (right) *insulator* is defined in the abstract; left insulator is similarly defined. If every nonzero element of R has a (right) insulator, then R is (right) strongly prime [4], and is obviously prime. Strongly prime rings are studied in [8], [9] under the name ATF, but detailed descriptions are given in [4], [5].

We note that if R is right strongly prime, then the right singular ideal ($Z(R)$) is zero, and if $R \subset S \subset Q(R)$, then S is right strongly prime and $Q(S) = Q(R)$ is simple [4, Proposition IV.1] and [9, Proposition 1.8].

Q_R will indicate Q considered as a right R-module. We wish to determine when Q_R is projective, to answer a question of Viola-Prioli [9, Question 5].

Lemma 1 [6, Exercises 1, 2, p. 86]. *A right R-module M is projective if and only if there exist $\{f_i\}_{i\in I} \subset \operatorname{Hom}(M, R)$ such that $f_i(m) = 0$ for almost all i, and there exists $\{m_i\}_{i \in I} \subset M$ such that $m = \sum_{i\in I} m_i f_i(m)$ for all m in M. Further, M is finitely generated projective if the I can be chosen finite.*

Lemma 2. *If S is a ring such that $R \subset S \subset Q(R)$, and S_R is projective*

Presented to the Society, April 23, 1974; received by the editors March 22, 1974 and, in revised form, June 24, 1974.

AMS (MOS) subject classifications (1970). Primary 16A08, 16A50; Secondary 16A12, 16A42, 16A52.

Key words and phrases. Ring of quotients, projective, injective, torsionless, nonsingular, prime ring, regular ring, simple ring.

(as a right R-module), then S_R is finitely generated.

Proof. Let $f \in \text{Hom}(S_R, R)$. If $f(1) = 0$, then for all s in S,

$$f(s) \cdot s^{-1}R = f(s \cdot (s^{-1}R)) = f(1) \cdot (s(s^{-1}R)) = 0.$$

Since $s^{-1}R$ is a dense right ideal of R and $f(s) \in R$, we have $f(1) = 0$ implies $f = 0$. If I is the index set of the preceding lemma, set $J = \{i \in I \mid f_i(1) \neq 0\}$. J is finite, and if $i \notin J$, $f_i = 0$; so J may take the place of I, and thus S_R is finitely generated.

Lemma 3 (e.g. [9, Lemma 2.1]). *If E is an injective module and M is nonsingular, then any homomorphism $f: E \longrightarrow M$ splits (that is, $E \simeq \ker f \oplus \text{im} f$ via f, so $\ker f$ and $\text{im} f$ are injective).*

Theorem 4. *If R is nonsingular, then Q_R is projective if and only if there exists $e = e^2 \in R$ such that $QeQ = Q$ and $eQ \subset R$. Further, for this e, $R \neq Q$ if and only if $ReR \neq R$.*

Proof. Suppose Q_R is projective; by Lemma 2, Q_R is finitely generated, hence there exist $\{f_i\}_{i=1}^n \subset \text{Hom}(Q_R, R)$ and $\{q_i\}_{i=1}^n \subset Q$, such that $q = \sum_{i=1}^n q_i f_i(q)$ for all q in Q. Since Q_R is injective and R is nonsingular, we may apply Lemma 3; thus $\text{im} f_i$ is injective. Therefore $\text{im} f_i = e_i R$ for some idempotent e_i in R. Now $e_i Q$ is always an essential extension of $e_i R$; hence $e_i R = e_i Q$. $\sum e_i R = \sum e_i Q$ is a finitely generated right Q-ideal, so there exists $e^2 = e \in Q$ such that $\sum e_i Q = eQ$, and obviously $eQ \subset R$. Finally, $1 \doteq \sum q_i f_i(1) \in QeQ$, so $QeQ = Q$.

If $eQ \subset R$ and $QeQ = Q$, there exist $q_i, p_i \in Q$ such that $1 = \sum q_i e p_i$. Define $f_i: Q_R \longrightarrow R$ by $f_i(q) = ep_i q \in eQ \subset R$. Then, $q = \sum q_i f_i(q)$, so Q is a finitely generated projective right R-module.

If $ReR \neq R$, then as $QeQ = Q$, clearly $R \neq Q$.

If $ReR = R$, R is Morita equivalent to a self-injective ring $eRe = eQe$, so R is self-injective, and thus $R = Q$. □

Theorem 4 can be restated as: R contains an injective right ideal J such that $\text{End}(J_R)$ is Morita equivalent to Q, or that R contains a nonzero right ideal of Q, J, such that $QJ = Q$.

Observe that in the proof we showed that the number of generators of Q_R is the size of the left insulator in Q chosen for e. We can even have the insulator of size 1 (so Q_R is cyclic) and $R \neq Q$, as will be shown.

Lemma 5. *If A is a subset of R, then A^r is an essential R-submodule of the right annihilator in Q of A.*

Proof. This may be deduced from [6, Exercise 6, p. 100]. □

Thus if a belongs to R, a right insulator for a in R is a right insulator for a in Q.

A ring is a (right) *F-ring* if every proper finitely generated left ideal has nonzero right annihilator. Obviously, all regular rings are two-sided F-rings.

Lemma 6. *If R is a right F-ring, then $r \in R$ has a right insulator in R if and only if $RrR = R$.*

Proof. If $\{r_i\}_{i=1}^n$ is a right insulator for r, then $\Sigma_{i=1}^n Rrr_i$ is a finitely generated left ideal with zero right annihilator. So $\Sigma Rrr_i = R$, thus $RrR = R$. Conversely, if $1 = \Sigma t_i rr_i$, then $\{r_i\}$ is a right insulator for r.

Theorem 4′. *If R is nonsingular, then Q_R is projective if and only if R contains an injective right ideal, eR for some $e = e^2 \in R$, such that e has a right insulator in R. For this e, $R = Q$ if and only if e has a left insulator in R.*

Proof. Suppose Q_R is projective. Then by Theorem 4, there exists $eQ = eR \subset R$ such that (i) eR is injective, and (ii) $QeQ = Q$. From (ii), e has a right insulator, $\{q_i\}$, in Q; clearly $\{eq_i\}$ is also a right insulator for e, and as $eQ \subset R$, it is an insulator for e in R.

Assume the converse condition. Since eR is injective, $eR = eQ$ (as in the proof of Theorem 4); since e has a right insulator in R, by Lemma 5, it has a right insulator in Q. Now Q is regular, so by Lemma 6, $QeQ = Q$; thus e satisfies the conditions of Theorem 4 and Q_R is projective.

If this e has a left insulator in r, $\{r_i\}$, then $\Sigma r_i eR = \Sigma r_i eQ$ is a finitely generated ideal of the regular ring Q, so there exists $f^2 = f \in Q$ such that $fQ = \Sigma r_i eQ$. Clearly, $fQ \subset R$, and as $\{r_i e\}^1 = (0)$, we must have $f^1 = (0)$; thus $f = 1$, and Q $(= fQ) = R$.

If $R = Q$, R is regular, so if e has a right insulator, by Lemma 6, it has a left insulator.

Corollary 7. *If Q is simple, then Q_R is projective and $R \neq Q$ if and only if there exists an injective right ideal of R, J, such that $(0) \neq RJ \neq R$.*

For instance if R is right strongly prime, Q is simple. Necessary and sufficient conditions for Q to be simple will be given (with examples) in [3].

Corollary 8. *If Q_R is projective and R satisfies any* one *of the following conditions:*

(i) *R is left strongly prime and right nonsingular,*

(ii) *R is a right F-ring and right nonsingular,*

(iii) *R is commutative semiprime,*

then $R = Q$.

Proof. (i) is immediate from Theorem 4′.

(ii) By Lemma 6, if $e \in R$ has a right insulator, it also has a left insulator.

(iii) For a commutative ring, semiprime is equivalent to nonsingular. □

Condition (ii) above generalizes Theorem 2.1 of [10].

Corollary 9. *If* $Z(R) = (0)$ *and* Q_R *is a projective module, and* $R \subset S \subset Q$, *then* Q_S *is projective.*

Proof. Clearly $Q(R) = Q(S)$; as $eR = eQ$, we have $eS = eQ$.

To obtain interesting examples of Q_R projective, we consider the following

Lemma 10. *If* T *is an overring of* R, *and* $a \in R$ *has no left insulator in* R, *then* a *has no left insulator in* $S = \langle aT, R\rangle$, *the subring of* T *generated by* aT *and* R.

Proof. Since aT is a right R-module, every element of S can be written in the form: $\Sigma\, r_i a t_i + r$; $t_i \in T$; $r, r_i \in R$. Suppose that $\{f_k = \Sigma_{jk} r_{jk} a t_{jk} + r_k\}$ is a finite subset of S. Since a has no left insulator in R, the finite set $\{r_k, r_{jk}\}_{jk,k}$ is not a left insulator for a; hence there exists nonzero u in R such that $ur_k a = ur_{jk} a = 0$ for all k, jk. It is clear that $uf_k a = 0$ for all k, so $\{f_k\}$ is not a left insulator for a in S.

Corollary 11. *If* $a \in R$ *has no left insulator in* R, *and* Q *is simple, then the ring* $S = \langle aQ, R\rangle$ *has* $Q(S) = Q$, $S \neq Q$, *and if* a *is nonzero,* Q_S *is a finitely generated projective right* S*-module. Further if* R *is prime (right strongly prime) then* S *is prime (right strongly prime).*

Proof. Follows immediately from Corollary 7 and the preceding lemma. □

If Q is simple self-injective and $e = e^2 \in Q$, then $R = eQ + Q(1 - e)$ has $Q = Q(R)$ (both on the right and the left) and Q is projective. $eQ(1 - e)$ is a nilpotent ideal of R. If, in particular, $Q = M_2 D$ where D is a division ring, then we obtain the overworked example:

$$R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \middle| \, a, b, c, \in D \right\}.$$

We now construct a right strongly prime (and hence right nonsingular prime) ring, S, such that $Q(S)$ is a cyclic projective right S-module. This answers in the negative two questions of Viola-Prioli [9, Questions 5 and 6].

Let F be a field, and $D = F[x, y]$, the free noncommuting algebra on two variables. D is a domain, so is right strongly prime. Further $DxD = \bigoplus_{i=0}^{\infty} y^i xD \simeq \bigoplus^{\aleph_0} D_D$; i.e., DxD is a countably generated free right D-module. Now, $DxD^l = (0)$, so the map (of rings)

$$D \longrightarrow \operatorname{End}(DxD_D), \qquad r \mapsto \hat{r}; \qquad \hat{r}(d) = rd$$

is a monomorphism, and $R = \operatorname{End}(DxD)$ is an essential extension of D as a right D-module, so $D \subset R \subset Q(D) = Q(R)$. Hence R is right strongly prime. But $R \simeq \operatorname{End}(\bigoplus D_D)$, the ring of $\aleph_0 \times \aleph_0$ matrices over D, with finitely many nonzero entries in each column. We show R has a proper torsion ideal [7, Chapter 2] on the left.

Let $r \in R$, write $r = (r_{ij})$, where r_{ij} is the i, j entry on the $\aleph_0 \times \aleph_0$ matrix that r represents, and let e_i be the matrix with a 1 in the i, i position and 0 elsewhere. Then $K = \bigoplus_{i=1}^{\infty} e_i R$ is a two-sided ideal and $K = \{r \in R \mid$ there exists N_r such that $i \geq N_r$ implies $r_{ij} = 0\}$ (each element of K contains only finitely many rows with nonzero entries).

Claim. R/K is flat as a right R-module.

In view of the proof of [7, Proposition 2.3], it suffices to show, given $k \in K$, there exists $k' \in K$ such that $k'k = k$. If $k = (k_{ij}) \in K$, there exists N_k such that $i \geq N_k$ implies $k_{ij} = 0$. Set $k' = \Sigma_{i=1}^{N_k} e_i \in K$; then $k'k = k$.

Since R/K is right flat, it follows from [7, Proposition 2.3], that K is a left torsion ideal. Since the corresponding torsion theory is closed under (in particular) submodules, direct sums and factors, it follows that no element of a proper torsion ideal may have a left insulator; in particular, e_1 has no left insulator in R. Since D is a domain and thus strongly prime, $Q(D) = Q(R)$ is simple; set $S = \langle e_1 Q, R \rangle$. By Corollary 11, $Q(S) = Q(D)$ is a finitely generated projective right S-module, S is right strongly prime (hence prime nonsingular) and $S \neq Q(S)$.

We note that since D is a domain, every element in $Q(D) = Q(S)$ has a one-element insulator: choose nonzero q in Q; there exists $d \in D$ such that qd is nonzero and belongs to D. Therefore $(qd)^r$ is zero, so the right annihilator in Q of qd is zero by Lemma 5. So $\{d\}$ is the desired insulator. Since Q is regular, there exists $u, v \in Q$ such that $ue_1 v = 1$; hence $Q = uQ = ue_1 Q = ue_1 S$; thus Q is a cyclic right S-module. □

A module M is *torsionless* if there exists a monomorphism $M \longrightarrow \pi R$. Obviously a projective module is torsionless.

In [1], conditions for a nonsingular semiprime ring to have Q torsionless are given. As noted in [1], for any ring, the sum of right ideals which are injective as right modules is a two-sided ideal. We say a two-sided ideal is right insulated if it contains a finite set whose right annihilator is zero.

Theorem 12. *If $Z(R) = (0)$, then Q_R is torsionless if and only if the sum of the injective right ideals is a faithful right R-module. Further $R = Q$ if and only if the sum of the injective right ideals is left insulated.*

Proof. As Q_R is torsionless, for all q in Q, there exists $f_q \in \operatorname{Hom}(Q_R, R)$ such that $f_q(q) \neq 0$. By Lemma 3, $f_q(Q)$ is an injective right ideal, and $\Sigma_{q \in Q} f_q(Q)$ is obviously faithful.

If $\{e_j R\}$ are injective and their sum is faithful, then $\pi e_j R$ is a faithful module. Hence there exists a monomorphism $R \longrightarrow \pi e_j R$; as $\pi e_j R$ is injective and Q is the injective hull of R, there exists a monomorphism $Q \to \pi e_j R \subset \pi R$.

If $R = Q$, 1 belongs to the sum of the injective right ideals.

If $R \neq Q$, proceed as in Theorem 4'.

Corollary 13. *For $Z(R) = (0)$, Q_R is torsionless if R contains a faithful injective right ideal.*

The condition of Corollary 13 is sufficient but not necessary: Put $Q = \pi F_j$ (an infinite product of fields), and $R = \langle \bigoplus F_j, 1\rangle$. Then $Q_R = Q(R)_R$ is torsionless, but R contains no faithful injective right ideals.

Corollary 14. *If Q is prime regular, then Q_R is torsionless if and only if R contains a nonzero injective right ideal.*

Corollary 15. *If $Z(R) = (0)$, and Q_R is torsionless, then Q is an essential extension of R as a left R-module.*

Proof. Let K be the sum of the injective right ideals of R, and q a nonzero element of Q. Then $Kq \neq (0)$ as K is faithful, and $Kq \subset K$ since any injective right ideal of R is of the form $eR = eQ$. Hence $(0) \neq Kq \subset K \subset R$.

Corollary 16 [1, Theorem 4]. *If R is semiprime, right nonsingular, then Q_R is torsionless if and only if the sum of the injective right ideals is essential as a right ideal.*

Proof. In a semiprime ring, a two-sided ideal is (right) faithful if and only if it is essential as a right ideal.

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