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THE AUTOMORPHISM GROUP OF THE TITS SIMPLE GROUP ${}^2F_4(2)'$

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ABSTRACT. In this paper, we determine the automorphism groups of the finite groups ${}^2F_4(2)$ and ${}^2F_4(2)'$. A consequence is that all Aut (G) are known, where G is a nonabelian composition factor of a finite group of Lie type.

The Tits simple group ${}^2F_4(2)'$ is a subgroup of index 2 in a finite group of Lie type ${}^2F_4(2)$. In [6], Ree constructs the groups ${}^2F_4(2^{2n+1})$ for $n \geq 0$ and determines their automorphism groups for $n \geq 1$, omitting the case $n = 0$. All other groups of Lie type and their nonabelian simple composition factors have had their automorphism groups determined in the literature. For Chevalley groups and "Steinberg variations," Steinberg [8] determines the automorphism groups by a uniform method. For the automorphism groups of the Suzuki groups (${}^2B_2(2^{2n+1})$, in Lie type notation), see Suzuki [9]. In [6] and [7], Ree determines the automorphism groups of the "Ree groups" of type F_4 and G_2 over all fields but the prime fields. However, since ${}^2G_2(3) \cong \mathrm{Aut}(\mathrm{SL}(2, 8))$ [5], this group is complete and is the automorphism group of its commutator subgroup by known results. So we turn to the two untreated cases: ${}^2F_4(2)$ and ${}^2F_4(2)'$.

Theorem 1. $\mathrm{Aut}({}^2F_4(2)) \cong {}^2F_4(2)$.

Theorem 2. $\mathrm{Aut}({}^2F_4(2)') \cong {}^2F_4(2)$.

We let $T = {}^2F_4(2)'$ and $F = {}^2F_4(2)$. We shall need the following properties of T and F:

(i) T is simple, has index 2 in F, and is the only nontrivial normal subgroup of F.

(ii) $|T| = 2^{11}3^3 5^2 13$.

(iii) Let $T_5 \in \mathrm{Syl}_5(T)$. Then T_5 is elementary abelian and is a TI-set in T. Also, $C_F(T_5) = T_5$ and $N_T(T_5)/T_5 \cong \mathbf{Z}_4 \circ \mathrm{SL}(2, 3)$.

(iv) Let $T_3 \in \mathrm{Syl}_3(T)$. Then T_3 is nonabelian of exponent 3. Also,

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$N_T(\mathbf{Z}(T_3)) = N_T(T_3) = T_3D$, where D is dihedral of order 8 and acts faithfully on T_3.

(v) All elements of order 3 in T are conjugate in T.

(vi) If T_3 and T_5 are as above, then $T = \langle N_T(T_5), T_3\rangle$.

(vii) If $T_2 \in \mathrm{Syl}_2(F)$, then $Z(T_2) = \langle z\rangle$ has order 2 and $C = C_F(z)$ is a maximal subgroup of order $2^{12}5$. If $Q = O_2(C)$, then Q has class 3, Q' is abelian, $|Q| = 2^{10}$, $|Q'| = 2^5$ and $\langle z\rangle = [Q', Q]$. An element h of order 5 in C operates nontrivially on Q/Q' and on $Q'/\langle z\rangle$; $C_Q(h)$ is cyclic of order 4 and $[Q, C_Q(h)] = Q'$; also, C/Q is a Frobenius group of order 20.

For property (i), see [10]; for (ii), [6]; for (vii), [2] or [4]. From the character table of T [3], we can get (iv), (v) and all of (iii) except $C_F(T_5) = T_5$; we do get $C_T(T_5) = T_5$. Since $|F:T| = 2$, if $C_F(T_5) > T_5$ there is an involution $j \in F\backslash T$ centralizing T_5. If we take h as in (vii), we get $C_F(h) \cong \langle h\rangle \times H$, where H is a Frobenius group of order 20 [2]. Taking $T_5 \geq \langle h\rangle$, we get $C_F(T_5) \leq C_F(h)$, and it is now clear that j cannot exist. As for (vi), we set $U = \langle N_T(T_5), T_3\rangle$, and suppose $U \neq T$. By (ii) and (iii), $|T:U| \equiv 1 \pmod{5^2}$ and $|T:U| \mid 2^7 13$. Thus, $|T:U| = 26$, and, by (iii), the natural action of T on the right cosets of U is doubly transitive. Hence a two-point stablizer V in this action has order $|U|/5^2 = 2^{10}3^3$. Since centralizers of elements of order 3 in T are 3-closed ((iv), (v)) and a Sylow 3-subgroup of V is not cyclic, a Sylow 3-subgroup of V centralizes $O_2(V)$ (use 5.3.16 of [1]). Since V is solvable, constraint implies $O_3(V) \neq 1$. Hence $|V| \leq 4|N_T(Z(T_3))| < 2^{10}3^3$ by (iv) and (v), a contradiction. Thus, (vi) holds.

We first prove Theorem 1. By referring to one of these arguments, we reduce the proof of Theorem 2 to quoting Theorem 1.

Let $a \in \mathrm{Aut}(F)$; we will show that a is an inner automorphism. Since F has trivial center, by (i), we may regard F as a subgroup of $\mathrm{Aut}(F)$. Set $G = \langle F, a\rangle$. In working for a contradiction, we may assume $a \notin F$ but $a^p \in F$, p a prime. Let C, Q z be as in (vii). By a Frattini-like argument, we may assume a centralizes z and also that a is trivial on C/Q, a Frobenius group of order 20 (a complete group). Another Frattini argument enables us to assume a centralizes h, $\langle h\rangle \in \mathrm{Syl}_5(C)$. By (vii), $C_Q(h) = \langle u\rangle \cong \mathbf{Z}_4$ and $[Q, u] = Q'$. Set $Q_0 = [Q, h]$. By (vii), $Q_0 \geq Q'$ and $Q_0 \cap Q'\langle u\rangle = Q'$. By (vii) again, $\mathrm{Hom}_{C/Q}(Q/Q', Q'/\langle z\rangle) \cong \mathbf{Z}_2$, and since u induces a nontrivial homomorphism, we may assume a corresponds to the trivial homomorphism, i.e., $[Q, a] \leq \langle z\rangle$. The three subgroups lemma then implies $[Q_0, a] = [Q, h, a] = 1$. Therefore, $C_Q(a) = Q$ or Q_0. Since a normalizes $N_C(\langle h\rangle)$ and satisfies $[N_C(\langle h\rangle), a] \leq N_Q(\langle h\rangle) = C_Q(h) = \langle u\rangle$, a normalizes $\langle u, v\rangle \in \mathrm{Syl}_2(N_C(\langle h\rangle))$. We expand $\langle h\rangle$ to $T_5 \in \mathrm{Syl}_5(F)$. Then $N_G(\langle h\rangle) \leq N_G(T_5)$, since T_5 is a TI-set in F (or G).

We first show that $p = 2$. Assume p is odd. Set $N_1 = N_G(T_5)$, $C_1 = C_G(T_5)$. Then N_1/C_1 is isomorphic to a subgroup of GL(2, 5) and has $N_1 \cap T/C_1 \cap T$ as a normal subgroup of index 2 or $2p$. From the structure of $N_1 \cap T/C_1 \cap T$ given in (iii), it is easy to verify that it has index a power of 2 in its GL(2, 5)-normalizer. Thus, $N_1 = C_1(N_1 \cap F)$, whence $|C_1| = |C_1 \cap T|p = 5^2 p$, by (iii). Clearly C_1 is abelian and admits N_1. Since G/T is abelian, $[N_1, C_1] \leq C_1 \cap T$. By (iii), $[N_1, C_1] = T_5$. Let $B = C_{C_1}(N_1)$. Then $C_1 = T_5 \times B$, by Fitting's lemma if $p \neq 3$ and by the abelianness of C_1 if $p \neq 5$. Thus, $|B| = p$ and $C_T(B) \geq N_T(T_5)$. Now $[F, B] \leq T$, which implies that $C_F(B) \geq N_F(T_5)$, since $|F: T| = 2$ and p is odd. Let X be a Sylow 3-subgroup of $N_T(T_5)$ and Y a Sylow 2-subgroup of $N_T(T_5) \cap C(X)$. By (iii), $|X| = 3$ and $Y \cong \mathbf{Z}_4$. Since $B \leq C(X)$, B normalizes $N_T(X)$. By (iv) and (v), $N_T(X) = T_3 D$, where $T_3 \in \mathrm{Syl}_3(T)$. Since D acts faithfully on T_3 and $Y \leq N_T(X)$, Y acts irreducibly on $T_3/Z(T_3)$. Since B centralizes Y and $|T_3/Z(T_3)| = 9$, it follows that B centralizes $T_3/Z(T_3)$. Hence B induces inner automorphisms on T_3, and so $C_{T_3}(B) > Z(T_3)$. Since Y normalizes $C_{T_3}(B)$, B centralizes T_3. Hence by (vi) and the Frattini argument, B centralizes F, a contradiction.

We have $p = 2$. If a were trivial on $T_5/\langle h \rangle$, then a would centralize T_5 and so $C_F(a) \geq \langle T_5, Q_0 \rangle = L$. But, $N_F(L) \geq \langle Q_0, h, u, v \rangle = C$, and so $N_F(L) = F$, or $L = C_F(a) = F$, a contradiction. Hence, a inverts $T_5/\langle h \rangle$. But now $h \in C$ and $|C|_5 = 5$ imply that z inverts $T_5/\langle h \rangle$. We simply replace a with az and obtain a similar contradiction. This completes the proof of Theorem 1.

To prove Theorem 2, we embed T in Aut(T) as usual and consider a subgroup P of Aut(T) which contains T with index p, a prime. By imitating the above arguments, we get a contradiction in case p is odd. Therefore Aut$(T)/T$ is a 2-group. By Theorem 1, F/T is embedded as a self-normalizing subgroup of Aut$(T)/T$. At once this gives $F/T \cong$ Aut$(T)/T$ and $F \cong$ Aut(T), as required to prove Theorem 2.

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