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ON FUNCTIONAL EQUATIONS RELATED TO MIELNIK'S PROBABILITY SPACES

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ABSTRACT. It is shown that the method used by C. V. Stanojevic to obtain a characterization of inner product spaces in terms of a Mielnik probability space of dimension 2 does not admit a generalization to dimension $n > 2$.

Let $f: [0, 2] \to [0, 1]$ be continuous and strictly increasing with $f(0) = 0$ and $f(2) = 1$. The class of all such functions f will be denoted by F. Likewise, let $g: [0, 2] \to [0, 2]$ be continuous but strictly decreasing with $g(0) = 2$ and $g(2) = 0$. Similarly, the class of all such functions g will be denoted by G. In [1] it is proved that the functional equation

$$(*) \qquad f + f \circ g = 1$$

where $(f \circ g)(t) = f[g(t)]$ has a solution $f \in F$ if and only if $g \in G$ is an involution, i.e., $g \circ g = e$ where e is the identity function on $[0, 2]$. Using this result it is also shown that a normed real linear space N is an inner product space if and only if for some $f \in F$, $(S, f(|x + y|))$ is a Mielnik probability space [2] of dimension 2. The functional equation $(*)$ served as a tool to obtain a new characterization of inner product spaces. In this note we consider the possibility of extending this characterization of inner product spaces to the case where p is a probability function generated by an appropriate function f and (S, p) is of dimension > 2.

Let $g^{(m)}$ denote m iterations of a function $g: I \to I$ where I is some interval. Also, suppose $g^{(n)} = e$ where e is the identity function on I and n is some positive integer. We shall show that the generalized functional equation

$$(**) \qquad f + f \circ g + f \circ g^{(2)} + \cdots + f \circ g^{(n-1)} = 1$$

(where f and g are functions belonging to a suitable generalization of the classes F and G defined earlier) collapses. In other words, the method from [1] cannot be extended in a straightforward manner to the case when (S, p) is of dimension >2. The following theorem (for a similar result for homeomorphisms see [3]) is the key to our result:

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Theorem. *Let $h: I \longrightarrow I$ be a function where I is an interval. If h is continuous and if for some $m \geq 2$, $h^{(m)} = e$, then h is an involution, i.e., $h \circ h = e$.*

Proof. Let $h(x) = h(y)$. Then, since $h^{(m)} = e$, we have $h^{(m)}(x) = h^{(m)}(y)$ implies $x = y$ and thus h is one-to-one. Hence, since h is continuous, h is strictly monotone. First we consider the case where h is strictly increasing. Then from $h(x) > x$ it follows that $x = h^{(m)}(x) > h^{(m-1)}(x) > \cdots > h(x) > x$ which is a contradiction. The contradiction also follows from the assumption $h(x) < x$. Hence $h(x) = x$ for all x in I and $h \circ h = e$. Next we consider the case where h is strictly decreasing. If $x < y$, then $h(x) > h(y)$ and $h^{(2)}(x) < h^{(2)}(y)$. Hence $h^{(2)}$ is strictly increasing. But $(h^{(2)})^{(m)} = (h^{(m)})^{(2)} = e^{(2)} = e$. Applying the first case to $h^{(2)}$ we get $h^{(2)} = e$. Therefore $h^{(2)} = e$ and h is an involution.

In particular, our theorem shows that the function $g: I \longrightarrow I$ appearing in our generalized functional equation (**) must be an involution. Thus (**) becomes $n(f + f \circ g)/2 = 1$ for n even and $(n+1)f/2 + ((n-1)/2)f \circ g = 1$ for n odd. Now if we want to extend the result from [1] to the n-dimensional case we have to have (**) since it is equivalent to Axiom (C) of Mielnik [2]. This shows that there is not a trivial extension to dimension n using the procedure from [1].

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