PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

ON FUNCTIONAL EQUATIONS RELATED TO MIELNIK'S PROBABILITY SPACES

C. F. BLAKEMORE AND C. V. STANOJEVIC

ABSTRACT. It is shown that the method used by C. V. Stanojevic to obtain a characterization of inner product spaces in terms of a Mielnik probability space of dimension 2 does not admit a generalization to dimension $n > 2$.

Let f: $[0, 2] \longrightarrow [0, 1]$ be continuous and strictly increasing with $f(0) = 0$ and $f(2) = 1$. The class of all such functions f will be denoted by F. Likewise, let g: $[0, 2] \longrightarrow [0, 2]$ be continuous but strictly decreasing with $g(0) = 2$ and $g(2) = 0$. Similarly, the class of all such functions g will be denoted by G. In [1] it is proved that the functional equation

$$(*) \qquad f + f \circ g = 1$$

where $(f \circ g)(t) = f[g(t)]$ has a solution $f \in F$ if and only if $g \in G$ is an involution, i.e., $g \circ g = e$ where e is the identity function on $[0, 2]$. Using this result it is also shown that a normed real linear space N is an inner product space if and only if for some $f \in F$, $(S, f(|x + y|))$ is a Mielnik probability space [2] of dimension 2. The functional equation $(*)$ served as a tool to obtain a new characterization of inner product spaces. In this note we consider the possibility of extending this characterization of inner product spaces to the case where p is a probability function generated by an appropriate function f and (S, p) is of dimension > 2.

Let $g^{(m)}$ denote m iterations of a function g: $I \longrightarrow I$ where I is some interval. Also, suppose $g^{(n)} = e$ where e is the identity function on I and n is some positive integer. We shall show that the generalized functional equation

$$(**) \qquad f + f \circ g + f \circ g^{(2)} + \cdots + f \circ g^{(n-1)} = 1$$

(where f and g are functions belonging to a suitable generalization of the classes F and G defined earlier) collapses. In other words, the method from [1] cannot be extended in a straightforward manner to the case when (S, p) is of dimension >2. The following theorem (for a similar result for homeomorphisms see [3]) is the key to our result:

Received by the editors July 16, 1974.

AMS (MOS) subject classifications (1970). Primary 39A15, 26A18; Secondary 46C10.

Key words and phrases. Functional equations, Mielnik probability spaces.

Theorem. *Let $h: I \longrightarrow I$ be a function where I is an interval. If h is continuous and if for some $m \geq 2$, $h^{(m)} = e$, then h is an involution, i.e., $h \circ h = e$.*

Proof. Let $h(x) = h(y)$. Then, since $h^{(m)} = e$, we have $h^{(m)}(x) = h^{(m)}(y)$ implies $x = y$ and thus h is one-to-one. Hence, since h is continuous, h is strictly monotone. First we consider the case where h is strictly increasing. Then from $h(x) > x$ it follows that $x = h^{(m)}(x) > h^{(m-1)}(x) > \cdots > h(x) > x$ which is a contradiction. The contradiction also follows from the assumption $h(x) < x$. Hence $h(x) = x$ for all x in I and $h \circ h = e$. Next we consider the case where h is strictly decreasing. If $x < y$, then $h(x) > h(y)$ and $h^{(2)}(x) < h^{(2)}(y)$. Hence $h^{(2)}$ is strictly increasing. But $(h^{(2)})^{(m)} = (h^{(m)})^{(2)} = e^{(2)} = e$. Applying the first case to $h^{(2)}$ we get $h^{(2)} = e$. Therefore $h^{(2)} = e$ and h is an involution.

In particular, our theorem shows that the function $g: I \longrightarrow I$ appearing in our generalized functional equation $(**)$ must be an involution. Thus $(**)$ becomes $n(f + f \circ g)/2 = 1$ for n even and $(n + 1)f/2 + ((n - 1)/2)f \circ g = 1$ for n odd. Now if we want to extend the result from [1] to the n-dimensional case we have to have $(**)$ since it is equivalent to Axiom (C) of Mielnik [2]. This shows that there is not a trivial extension to dimension n using the procedure from [1].

REFERENCES

1. C. V. Stanojevic, *Mielnik's probability spaces and characterization of inner product spaces*, Trans. Amer. Math. Soc. **183** (1973), 441–448.

2. B. Mielnik, *Geometry of quantum states*, Comm. Math. Phys. **9** (1968), 55–80. MR **37** #7156.

3. N. McShane, *On the periodicity of homeomorphisms of the real line*, Amer. Math. Monthly **68** (1961), 562–563. MR **24** #A199.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW ORLEANS, NEW ORLEANS, LOUISIANA 70122

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI AT ROLLA, ROLLA, MISSOURI 65401