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THE FIXED POINT PROPERTY FOR HOMEOMORPHISMS OF 1-ARCWISE CONNECTED CONTINUA[1]

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ABSTRACT. It is shown that continua which are arcwise connected and contain no simple closed curves have the fixed point property for homeomorphisms, answering in the affirmative a question of Bing. The proof uses measure theoretic techniques. Given a homeomorphism h of a compact metric space X onto itself, a probability measure is constructed on X which is invariant under h.

Introduction. A continuum (compact, connected, metric space) X is said to be *1-arcwise connected* if given any two points $x, y \in X$, $x \neq y$, there is a unique arc in X whose endpoints are x and y. This is equivalent to saying that X is arcwise connected and contains no simple closed curves. It is well known that such spaces need not have the fixed point property (see [6, p. 884]). In [1, p. 126, Question 6] Bing asks whether such spaces have the fixed point property for homeomorphisms. The object of this paper is to answer this question in the affirmative. The proof uses measure theoretic techniques. The paper is divided into two sections. In the first, the requisite analysis is developed. The second section is devoted to the proof of the fixed point theorem.

1. If X is a continuum and μ is a (complete) regular Borel measure on X, then the μ-measurable subsets of X will always include the analytic sets (see [4, p. 482]). This section is devoted to showing that the arc components of any continuum are analytic (and hence μ-measurable) and to producing a probability measure on an arbitrary continuum X which is invariant under a given homeomorphism of X onto itself.

Definition 1.1. Let X be a compact metric space. Then 2^X will denote the space of all closed subsets of X with the Hausdorff metric (see [4, p. 407]) and $C(X)$ will denote the subspace of 2^X consisting of all subcontinua of X. 2^X is compact (see [5, pp. 45, 47]) and $C(X)$ is closed in 2^X (see [5, p. 139, Theorem 14]).

Definition 1.2. A subset A of a complete separable metric space is said to be *analytic* if it is the continuous image of a Borel subset of some complete

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separable metric space. This is equivalent to saying that A is a continuous image of the irrational numbers (see [4, p. 478]).

Theorem 1.3. *Let X be a compact metric space and let $\mathcal{B}$ be a Borel subset of 2^X. Then $\bigcup \mathcal{B}$ is analytic in X.*

Proof. Let $\mathcal{B}^*$ denote the following subset of $X \times 2^X$: $\{(x, B): B \in \mathcal{B}$ and $x \in B\}$. $\mathcal{B}^*$ is a Borel set since it is the intersection of the Borel set $X \times \mathcal{B}$ with the closed set $\{(x, A): A \in 2^X$ and $x \in A\}$. Let π be the left projection of $X \times 2^X$ onto X. Then $\pi(\mathcal{B}^*) = \bigcup \mathcal{B}$. Q.E.D.

Theorem 1.4. *Let X be a metric continuum, $p \in X$ and let $L_p(X)$ denote the family of all locally connected subcontinua of X containing p. Then $L_p(X)$ is a Borel set in 2^X.*

Proof. For each $k, n \in N$ let $F_{k,n}$ denote the family of all subsets A of X which contain p and can be written as a union of $\leq n$ subcontinua of X, each of diameter $\leq 1/k$. Each of the $F_{k,n}$'s is closed, so the set $F'_k = \bigcup_{n=1}^{\infty} F_{k,n}$ is an F_σ, $k = 1, 2, \cdots$. Let $F_k = F'_k \cap C(X)$, $k = 1, 2, \cdots$. Then each F_k is an F_σ and consists of all subcontinua of X which contain p and can be written as a union of finitely many subcontinua of X, each of diameter $\leq 1/k$. By [5, Theorem 2, p. 256] $L_p(X) = \bigcap_{k=1}^{\infty} F_k$. Q.E.D.

Definition 1.5. If X is a continuum and $p \in X$, then the *arc component of X generated by p* is $\{x \in X$: there is an arc in X whose endpoints are p and $x\} \cup \{p\}$. This set will be denoted by $A_p(X)$.

Theorem 1.6. *Let X be a continuum and $p \in X$. Then $A_p(X) = \bigcup L_p(X)$.*

Proof. This follows easily from the fact that all arcs are locally connected and all locally connected continua are arcwise connected. Q.E.D.

Corollary 1.7. *Let X be a continuum and $p \in X$. Then $A_p(X)$ is an analytic set.*

Theorem 1.8. *Let X be a compact metric space and $h: X \to X$ a homeomorphism. Then there is a nonnegative regular Borel measure μ on X such that $\mu(X) = 1$ and $\mu(A) = \mu(h^n(A))$ for every integer n and every μ-measurable set $A \subset X$.*

Proof.[2] The space $\mathfrak{M}$ of all (complex) regular Borel measures on X can be identified with the dual of the Banach space of all continuous, complex-valued functions on X (see [3, pp. 361–363]). The set P of all probability measures (nonnegative measures μ such that $\mu(X) = 1$) forms a closed

[2] The proof of this theorem is due to David Ragozin.

convex subset of the unit ball in $\mathfrak{M}$ which is compact in the weakest topology on $\mathfrak{M}$ which makes all of the evaluation functionals continuous (this is Alaoglu's theorem. See [2, Theorem 2, p. 424]).

The homeomorphism h and its (positive and negative) iterates together with the identity generate a commutative group of actions on $\mathfrak{M}$ given by the formulas $h_*^n(\mu)(A) = \mu(h^n(A))\ \forall \mu \in \mathfrak{M}, \forall n \in Z$ and for all μ-measurable subsets A of X. Each of the actions h_*^n is continuous, affine and carries P into itself. By the Markov-Kakutani theorem (see [2, Theorem 6, p. 456 and remark following]) there is a measure $\mu \in P$ which is fixed by all of the actions h_*^n. Thus for each $n \in Z$ and for each μ-measurable subset A of X $\mu(A) = h_*^n(\mu)(A) = \mu(h^n(A))$.

Corollary 1.9. *Let X be a compact metric space and $h: X \rightarrow X$ a homeomorphism. Then there is a complete, nonnegative, regular, Borel measure μ on X such that $\mu(X) = 1$ and $\mu(A) = \mu(h^n(A))$ for every integer n and every μ-measurable set $A \subset X$.*

Proof. Let μ' be as in 1.8 and let μ be the completion of μ'. Since every μ-measurable subset of X differs from a Borel set by a set E which is a subset of a Borel set of measure 0, it is clear that μ has the desired properties. Q.E.D.

2. Throughout this section X will denote a fixed 1-arcwise connected continuum and h a fixed point free homeomorphism of X onto itself. A series of lemmas will be proved, leading to a contradiction.

Definition 2.1. Given $x, y \in X, x \neq y$, $[x, y]$ will denote the unique arc in X whose endpoints are x and y. $[x, y)$, $(x, y]$ and (x, y) are defined analogously. A subset of X will be said to be *of the form* $[a, \infty)$ if it is the union of a nest of arcs $[a, x]$ in X and is not contained in any arc $[c, d]$ in X.

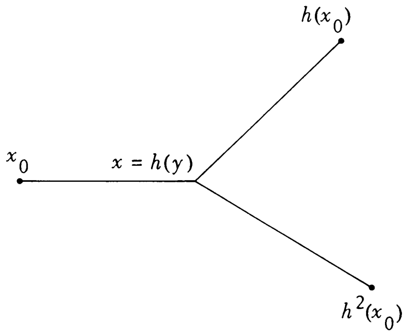
Note that if A and B are arcwise connected subsets of X and $x, y \in A \cap B$, then $[x, y] \subset A \cap B$. Thus the intersection of any two arcwise connected subsets of X is arcwise connected (in particular X contains no simple closed curves).

Lemma 2.2. *X contains a subset L which is a 1-1 continuous image of R and such that $h(L) = L$.*

Proof. Pick $x_0 \in X$. Consider the arc $[x_0, h(x_0)]$ and its image $[h(x_0), h^2(x_0)]$. If these meet only in $\{h(x_0)\}$, then

$$L = \cdots \cup [h^{-2}(x_0), h^{-1}(x_0)] \cup [h^{-1}(x_0), x_0] \cup [x_0, h(x_0)]$$
$$\cup [h(x_0), h^2(x_0)] \cup \cdots$$

will be the desired set (for each $n \in Z$ maps the interval $[n, n+1]$ onto the arc $[h^n(x_0), h^{n+1}(x_0)]$. The mapping will be 1-1 since X contains no simple closed curves). If $[x_0, h(x_0)] \cap [h(x_0), h^2(x_0)]$ contains more than $h(x_0)$, then the intersection will be an arc $[x, h(x_0)] = [h(x_0), h(y)]$ for some $x, y \in [x_0, h(x_0)]$ (see figure).



If $y \in [x, h(x_0)]$, then $x \in [x_0, y]$ which implies that

$$h(x) \in [h(x_0), h(y)] = [h(x_0), x] = [x, h(x_0)].$$

Since $y \in [x, h(x_0)]$, either $y \in [x, h(x)]$ or $h(x) \in [x, y]$. Thus either

$$[x, y] \subset [x, h(x)] = [h(x), h(y)] \quad \text{or} \quad [h(x), h(y)] = [x, h(x)] \subset [x, y].$$

In either case h would have a fixed point.

Thus $y \notin [x, h(x_0)]$. So it must be the case that $y \in [x_0, x]$. But then $x \in [y, h(x_0)]$ which implies that $h(x) \in [h(y), h^2(x_0)] = [x, h^2(x_0)]$. Thus

$$[h(y), h^2(y)] = [x, h(x)] \subset [x, h^2(x_0)].$$

Moreover, $[y, h(y)] = [y, x] \subset [x_0, x]$. Since $[x_0, x] \cap [x, h^2(x_0)] = \{x\}$, $[y, h(y)]$ and its image $[h(y), h^2(y)]$ can meet only in $\{h(y)\}$. Thus

$$L = \cdots \cup [h^{-2}(y), h^{-1}(y)] \cup [h^{-1}(y), y] \cup [y, h(y)] \cup [h(y), h^2(y)] \cup \cdots$$

will be the desired set. Q.E.D.

For the rest of this section L and y will be defined as in the proof of 2.2 (if $[x_0, h(x_0)] \cap [h(x_0), h^2(x_0)] = \{h(x_0)\}$, then take $y = x_0$).

Lemma 2.3. *L is not contained in any set of the form $[a, b)$ or $[a, \infty)$ in X.*

Proof. If this were the case, then one of the two sequences $\{h^n(y)\}_{n=1}^\infty$ and $\{h^{-n}(y)\}_{n=1}^\infty$ would converge "toward a" and would hence converge to some point of X which would be a fixed point for h. Q.E.D.

Lemma 2.4. *If $x \in X - L$, then there is an arc in X which contains x and meets L in a single point.*

Proof. Let $x \in X - L$ and $p \in L$. The set $[x, p] \cap L$ is arcwise connected. If it is compact, we are done. If not, then $[x, p] \cap L$ contains a "cofinal" subset of L and $[x, p] \cup L$ would contain an interval of the form $[a, \infty)$ or $[a, b)$ containing L. Q.E.D.

Thus if $x \in X - L$, there is a unique arc $[x, b]$ in X which meets L only in $\{b\}$. Call this arc $A(x)$.

Definition 2.5. Given any set $B \subset L$, let

$$A(B) = \{A(x)\colon\ x \in X - L \text{ and } A(x) \cap L \subset B\} \cup B.$$

Lemma 2.6. *Let μ be as in Corollary 1.9 and let $a, b \in L$, $a \neq b$. Then $A((a, b))$ is μ-measurable.*

Proof. Let $p \in (a, b)$ and for each $n \in N$ let U_n and V_n be disjoint neighborhoods of a and b, respectively, such that $p \notin U_n \cup V_n$ and U_n and V_n have diameters $< 1/n$. Let C_n be the union of all arcs in X emanating from p which do not meet $U_n \cup V_n$ and let $D_n = \mathrm{Cl}(C_n)$. Let E_n be the arc component of D_n generated by p.

Note that $\forall n \in N$ $C_n \subset E_n$ and E_n contains no points of $L - (a, b)$ (any arc in X containing p and a point of $L - (a, b)$ must pass through a or b). Thus $A((a, b)) = \bigcup_{n=1}^{\infty} E_n$ and by Corollary 1.7 the E_n's are all analytic and hence μ-measurable. Q.E.D.

Definition 2.7. For each $n \in Z$ let $A_n = A([h^n(y), h^{n+1}(y)))$. Note that the sets A_n are all disjoint and since h is a homeomorphism, $h(A_n) = A_{n+1}$ $\forall n \in Z$. Moreover, $X = \bigcup_{n=-\infty}^{\infty} A_n$.

Lemma 2.8. *Each of the sets A_n is μ-measurable (μ as in 1.9).*

Proof. Each A_n is of the form $A([a, b))$ for some $a, b \in L$. $A([a, b)) = A(\{a\}) \cup A((a, b))$, so in view of 2.6 it suffices to show that $A(\{a\})$ is μ-measurable.

Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be a nested sequence of subarcs of L whose intersection is $\{a\}$. Then $A(\{a\}) = \bigcap_{n=1}^{\infty} A((a_n, b_n))$, so by 2.6 $A(\{a\})$ is μ-measurable. Q.E.D.

Now let μ be as in 1.9. Then since each A_n is a (positive or negative) iterate of A_0 under h, 2.8 implies that $\mu(A_n) = \mu(A_0) \forall n \in Z$. Thus, in view of the remarks preceding 2.8, we have

$$1 = \mu(X) = \sum_{n=-\infty}^{\infty} \mu(A_n) = \sum_{n=-\infty}^{\infty} \mu(A_0) = 0 \quad \text{or} \quad \infty.$$

This contradiction establishes the fixed point theorem.

Theorem. *Let X be a 1-arcwise connected continuum and $h\colon X \to X$ a homeomorphism. Then h has a fixed point.*

Remarks. The above Theorem can be improved slightly. If h is 1-1 but not onto, then one can find a fixed point for h by first locating a 1-arcwise connected subcontinuum of X which h maps onto itself. The author would like to thank E. D. Tymchatyn and A. Lelek for a number of valuable conversations during the course of work on this paper.

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