NONDEGENERATE HIGHER DEGREE FORMS
OVER DEDEKIND DOMAINS

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ABSTRACT. It is determined which finitely generated projective modules over a Dedekind domain admit nondegenerate symmetric r-linear forms.

In [1] D. K. Harrison introduces a Grothendieck ring based on nondegenerate symmetric r-linear forms, \( r \geq 2 \), over a Noetherian commutative ring. The structure results are quite strong for \( r \geq 3 \).

In this note we determine which finitely generated projective modules over a Dedekind domain admit a nondegenerate r-linear form.

Definition. Let \( R \) be a commutative ring and let \( V \) denote a finitely generated \( R \)-module. A symmetric r-linear map \( \theta : V \rightarrow R \) is said to be nondegenerate if the induced group homomorphism \( \Gamma : V \otimes \cdots \otimes V \rightarrow \text{Hom}_R(V, R) \)
\[ \Gamma(x_1 \otimes \cdots \otimes x_{r-1})(x) = \theta(x_1, \ldots, x_{r-1}, x) \]
is surjective. In this case \((V, \theta)\) is called a nondegenerate space. Harrison's ring is the Grothendieck ring based on isomorphism classes of nondegenerate symmetric r-linear space under orthogonal direct sum and tensor product.

According to [1, Lemma 2.6], if \((V, \theta)\) is nondegenerate, then \( V \) is projective; so let \( R \) be a Dedekind domain and \( V \) a finitely generated projective \( R \)-module. Thus \( V \cong \mathcal{U} \otimes R \oplus \cdots \oplus R \) for some ideal \( \mathcal{U} \) of \( R \). If \( V \) admits a nondegenerate bilinear form, then \( \Gamma : V \rightarrow V^* \) induced by \( \theta \) is an isomorphism, so we see that \( \mathcal{U} \cong \mathcal{U}^{-1} \), i.e. \( \mathcal{U}^2 \) is principal. Conversely if \( \mathcal{U}^2 = \lambda R \) for some \( \lambda \) in the field of fractions \( K \) of \( R \), then it is easy to see that \( \theta(x, y) = xy/\lambda \) is a nondegenerate bilinear form on \( V \). Thus \( V \) admits a nondegenerate bilinear form if and only if \( \mathcal{U}^2 \) is principal. Similarly one argues that \( \mathcal{U} \) admits a nondegenerate r-linear form if and only if \( \mathcal{U}^r \) is principal.

Now consider the case \( r \geq 3 \) with rank \( V > 1 \). We claim that in this case \( V \) will always admit a nondegenerate form \( \theta \). It will suffice to consider \( V = \mathcal{U} \oplus R \). Let \( e_1, e_2 \in K \oplus K \) be such that \( V = \mathcal{U}e_1 + Re_2 \). A symmetric r-linear form on \( V \) corresponds precisely to elements \( \mu_0, \mu_1, \ldots, \mu_r \), where \( \mu_k \in \mathcal{U}^{-k}, k = 0, \ldots, r \) as follows: Given the \( \mu_k \) we define \( \theta \) on \( K \oplus K \) by \( \theta(e_{i_1}, \ldots, e_{i_r}) = \mu_k \) if exactly \( k \) of the subscripts \( i_1, \ldots, i_r \) are equal to 1. Then since \( \mu_k \in \mathcal{U}^{-k} \), \( \theta \) restricted to \( V^r \) gives a mapping from \( V^r \) into \( R \).

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Lemma. Let $\theta$ be a symmetric $r$-linear form on $\mathbb{U} \oplus \mathbb{R}$ associated with $\mu_0, \mu_1, \ldots, \mu_r$ as above. In order that $\theta$ be nondegenerate, it is necessary and sufficient that for any $\alpha \in \mathbb{U}^{-1}$, $x \in \mathbb{R}$, the equations

$$x_{r-1}\mu_r + x_{r-2}\mu_{r-1} + \cdots + x_0\mu_1 = \alpha,$$

$$(*)$$

$$x_{r-1}\mu_r + x_{r-2}\mu_{r-2} + \cdots + x_0\mu_0 = x$$

have simultaneous solutions with $x_j \in \mathbb{U}$, $j = 0, \ldots, r - 1$.

Proof. It follows directly from the definition that $\theta$ is nondegenerate iff given $\alpha \in \mathbb{U}^{-1}$, $x \in \mathbb{R}$ there are $a^1_1 \cdots a^1_{r-1} \in \mathbb{U}$, $z^1_1, \ldots, z^1_{r-1} \in \mathbb{R}$, $i = 1, \ldots, m$, such that

$$\left( \sum_{i=1}^{m} a^1_i \cdots a^1_{r-1} \right) \mu_r + \left( \sum_{j=1}^{r-1} \sum_{i=1}^{m} a^1_i \cdots \hat{a}^1_j \cdots a^1_{r-1} \right) z^1_j \mu_{r-j} + \cdots + \left( \sum_{j=1}^{r-1} \sum_{i=1}^{m} a^1_i \cdots \hat{a}^1_j \cdots \hat{a}^1_{r-j} \right) z^1_j \mu_{r-j}$$

and

$$\left( \sum_{i=1}^{m} a^t_i \cdots a^t_{r-1} \right) \mu_{r-1} + \left( \sum_{j=1}^{r-1} \sum_{i=1}^{m} a^t_i \cdots \hat{a}^t_j \cdots a^t_{r-1} \right) z^t_j \mu_{r-1-j} + \cdots + \left( \sum_{j=1}^{r-1} \sum_{i=1}^{m} z^t_i \cdots z^t_{r-1} \right) \mu_0 = x.$$

(A caret over a symbol indicates its deletion.) Thus necessity is clear.

Now suppose we have a solution $x_0, \ldots, x_{r-1}$ to $(*)$, with $x_j \in \mathbb{U}$, $j = 0, \ldots, r - 1$. We show by induction that there are positive integers $M_1 < M_2 < \cdots < M_{r+1}$, such that for each $t \leq r$ and each $s \leq t$,

$$\sum_{i=1}^{M_{t+1}} a^t_i \cdots \hat{a}^t_j \cdots \hat{a}^t_{r-1} \cdots a^t_{r-1} z^t_i \cdots z^t_{j-1} = x_{r-s-1}.$$

Choose $a^1_1, \ldots, a^1_{r-1} \in \mathbb{U}$, $i = 1, \ldots, M_1$, such that $\sum_{i=1}^{M_1} a^1_i \cdots a^1_{r-1} = x_{r-1}$. Choose $z^1_j = 0$ for $1 \leq j \leq r - 1$, $i = 1, \ldots, M_1$.

Now suppose $t \geq 1$ and $M_1 < M_2 < \cdots < M_t$ have been chosen along with $a^t_1, \ldots, a^t_{r-1}; z^t_1, \ldots, z^t_{r-1}$ such that $\sum_{i=1}^{M_t} a^t_i \cdots a^t_{r-1} = x_{r-1}$ and

$$\sum_{1 \leq k_1 < \cdots < k_{j-1} \leq r-1} a^t_{k_1} \cdots a^t_{k_{j-1}} z^t_{k_1} \cdots z^t_{k_{j-1}} = x_{r-j}$$

for $2 \leq j \leq t$. 

For $M_{t+1}$ choose

$$\sum_{i=1}^{M_{t+1}} a^t_i \cdots \hat{a}^t_j \cdots \hat{a}^t_{r-1} \cdots a^t_{r-1} z^t_i \cdots z^t_{j-1} = x_{r-s-1}.$$

Next choose

$$\sum_{i=1}^{M_{t+1}} a^{t+1}_i \cdots \hat{a}^{t+1}_j \cdots \hat{a}^{t+1}_{r-1} \cdots a^{t+1}_{r-1} z^{t+1}_i \cdots z^{t+1}_{j-1} = x_{r-(s+1)-1}.$$
Choose \( M_{t+1} > M_t \) and for \( i = M_t + 1, \ldots, M_{t+1} \), choose \( a^i_{t+1}, \ldots, a^i_{r-1} \) \( \in \mathbb{U} \) such that
\[
\sum_{i=M_{t+1}}^{M_{t+1}} a^i_{j+1} \cdots a^i_{r-1} = x_{r-t-1}.
\]
Let \( a^i_1 = \cdots = a^i_t = 0, z^i_1 = \cdots = z^i_t = 1, z^i_{t+1} = \cdots = z^i_{r-1} = 0 \) for \( M_t < i \leq M_{t+1} \). Then by induction,
\[
\sum_{i=1}^{M_{t+1}} a^i_1 \cdots a^i_{r-1} = \sum_{i=1}^{M_t} a^i_1 \cdots a^i_{r-1} = x_{r-1}
\]
and
\[
\sum_{j_1, \ldots, j_s \leq r-1} \sum_{i=1}^{M_t} a^i_{j_1} \cdots a^i_{j_2} \cdots a^i_{j_s} = x_{r-s-1} \quad \text{for } s < t.
\]
Finally,
\[
\sum_{j_1, \ldots, j_t} \sum_{i=1}^{M_{t+1}} a^i_{j_1} \cdots a^i_{j_2} \cdots a^i_{j_t} z^i_{j_1} \cdots z^i_{j_t} = \sum_{i=M_{t+1}}^{M_{t+1}} a^i_{t+1} \cdots a^i_{r-1} = x_{r-t-1}.
\]
Now the Lemma follows by induction, with \( m = M_r \). □

We now suppose that \( r \geq 3, \alpha \in \mathbb{U}^{-1} \) and \( x \in \mathbb{R} \), and show there are \( \mu_0, \ldots, \mu_r \) with \( \mu_i \in \mathbb{U}^{-i} \) such that (*) has a solution \( x_1, \ldots, x_{r-1} \) with \( x_i \in \mathbb{U}^i \). To this end choose \( \mu_0 = 1, \mu_1 = \cdots = \mu_{r-2} = 0; \) and choose \( \mu_{r-1} \in \mathbb{U}^{1-r} \) and \( \mu_r \in \mathbb{U}^{-r} \) such that
\[
\mu_r \mathbb{U}^{r-1} + \mu_{r-1} \mathbb{U}^{r-2} = \mathbb{U}^{-1}.
\]
We can do this since, in a Dedekind domain, any nonzero element of an ideal can be extended to a two element generating set for that ideal. Let \( x_{r-1} \in \mathbb{U}^{r-1} \), \( x_{r-2} \in \mathbb{U}^{r-2} \) be such that
\[
\mu_r x_{r-1} + \mu_{r-1} x_{r-2} = \alpha.
\]
Then let \( x_0 = x - \mu_{r-1} x_{r-1} \). It is clear that \( 0, \ldots, 0, x_{r-2}, x_{r-1} \) is a solution to (*).

We summarize these results as follows.
Theorem. Let $R$ be a Dedekind domain and let $V$ be a finitely generated projective $R$-module of rank $n \geq 1$, and let $r \geq 2$. Suppose $V \cong \mathcal{U} \oplus R \oplus \ldots \oplus R$ where $\mathcal{U}$ is a nonzero ideal in $R$. Then $V$ admits a nondegenerate symmetric $r$-linear form iff

1. $r = 2$ and $\mathcal{U}^2$ is principal, or
2. $r \geq 3$, $n = 1$ and $\mathcal{U}^r$ is principal, or
3. $r \geq 3$, $n \geq 2$.

Note that $\mu_0, \ldots, \mu_r$ give a nondegenerate form on $\mathcal{U} \oplus R$ iff the homomorphism from $\mathcal{U}^{r-1} \oplus \mathcal{U}^{r-2} \oplus \ldots \oplus \mathcal{U} \oplus R$ to $\mathcal{U}^{-1} \oplus R$ given by the matrix

\[
\begin{pmatrix}
\mu_r & \mu_{r-1} & \cdots & \mu_1 \\
\mu_{r-1} & \mu_{r-2} & \cdots & \mu_0
\end{pmatrix}
\]

is surjective.

The referee has pointed out that the Theorem applies in case $R = D[x]$, where $D$ is a Dedekind domain. See [2, Corollary (6.4)].

Finally, we answer a question raised by the referee, thanks to a conversation with Paul Eakin. Let $R$ be a Noetherian domain of Krull dimension $d$. If $\mathcal{U}$ is a projective ideal and if $r \geq d + 2$, then $\mathcal{U} \oplus R$ has a nondegenerate symmetric form. To see this it suffices to produce $\mu_r, \mu_{r-1}, \ldots, \mu_{r-d}$, with $\mu_j \in \mathcal{U}^{-j}$, such that $\mathcal{U}^r \mu_r + \cdots + \mathcal{U}^{r-d} \mu_{r-d} = R$. (For then these $\mu_j$, along with $\mu_1 = 0$ will give a solution for the first equation in (*), with $x_0$ arbitrary. Then with $\mu_0 = 1$, we can choose $x_0$ so that the second equation holds.)

Take $\mu_r = 1$ and choose $\mu_{r-1}$ so that $\mathcal{U}^{r-1} \mu_{r-1}$ is not contained in any of the finite number of minimal primes over $\mathcal{U}^r$. This is possible since $\mathcal{U}^{r-1}$ is invertible. Then $\mathcal{U}^r + \mathcal{U}^{r-1} \mu_{r-1}$ has height $\leq d - 1$. If $\mathcal{U}^r + \mathcal{U}^{r-1} \mu_{r-1} \neq R$, choose $\mu_{r-2}$ so that $\mathcal{U}^{r-2} \mu_{r-2}$ is not contained in any of the minimal primes over $\mathcal{U}^r + \mathcal{U}^{r-1} \mu_{r-1}$. Then $\mathcal{U}^r + \mathcal{U}^{r-1} \mu_{r-1} + \mathcal{U}^{r-2} \mu_{r-2}$ has height $\leq d - 2$. Continuing in this manner, we get the desired $\mu_r, \ldots, \mu_{r-d}$.

REFERENCES