PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

NONDEGENERATE HIGHER DEGREE FORMS OVER DEDEKIND DOMAINS

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ABSTRACT. It is determined which finitely generated projective modules over a Dedekind domain admit nondegenerate symmetric r-linear forms.

In [1] D. K. Harrison introduces a Grothendieck ring based on nondegenerate symmetric r-linear forms, $r \geq 2$, over a Noetherian commutative ring. The structure results are quite strong for $r \geq 3$.

In this note we determine which finitely generated projective modules over a Dedekind domain admit a nondegenerate r-linear form.

Definition. Let R be a commutative ring and let V denote a finitely generated R-module. A symmetric r-linear map $\theta : V^r \longrightarrow R$ is said to be *nondegenerate* if the induced group homomorphism $\Gamma : V \otimes \cdots \otimes V \longrightarrow \mathrm{Hom}_R(V, R) = V^*$ given by $\Gamma(x_1 \otimes \cdots \otimes x_{r-1})(x) = \theta(x_1, \ldots, x_{r-1}, x)$ is surjective. In this case (V, θ) is called a *nondegenerate space*. Harrison's ring is the Grothendieck ring based on isomorphism classes of nondegenerate symmetric r-linear space under orthogonal direct sum and tensor product.

According to [1, Lemma 2.6], if (V, θ) is nondegenerate, then V is projective; so let R be a Dedekind domain and V a finitely generated projective R-module. Thus $V \cong \mathfrak{A} \oplus R \oplus \cdots \oplus R$ for some ideal $\mathfrak{A}$ of R. If V admits a nondegenerate bilinear form, then $\Gamma : V \longrightarrow V^*$ induced by θ is an isomorphism, so we see that $\mathfrak{A} \cong \mathfrak{A}^{-1}$, i.e. $\mathfrak{A}^2$ is principal. Conversely if $\mathfrak{A}^2 = \lambda R$ for some λ in the field of fractions K of R, then it is easy to see that $\theta(x, y) = xy/\lambda$ is a nondegenerate bilinear form on $\mathfrak{A}$. Thus V admits a nondegenerate bilinear form if and only if $\mathfrak{A}^2$ is principal. Similarly one argues that $\mathfrak{A}$ admits a nondegenerate r-linear form iff $\mathfrak{A}^r$ is principal.

Now consider the case $r \geq 3$ with rank $V > 1$. We claim that in this case V will always admit a nondegenerate form θ. It will suffice to consider $V = \mathfrak{A} \oplus R$. Let $e_1, e_2 \in K \oplus K$ be such that $V = \mathfrak{A}e_1 + Re_2$. A symmetric r-linear form on V corresponds precisely to elements $\mu_0, \mu_1, \ldots, \mu_r$, where $\mu_k \in \mathfrak{A}^{-k}$, $k = 0, \ldots, r$ as follows: Given the μ_k we define θ on $K \oplus K$ by $\theta(e_{i_1}, \ldots, e_{i_r}) = \mu_k$ if exactly k of the subscripts $i_1, \ldots, i_r$ are equal to 1. Then since $\mu_k \in \mathfrak{A}^{-k}$, θ restricted to V^r gives a mapping from V^r into R.

Presented to the Society, January 17, 1974 under the title *Forms on Dedekind domains*; received by the editors July 29, 1974.

AMS (MOS) subject classifications (1970). Primary 16A50, 12A99.

Key words and phrases. Dedekind domain, projective module, nondegenerate form.

Lemma. *Let θ be a symmetric r-linear form on $\mathfrak{A} \oplus R$ associated with $\mu_0, \mu_1, \ldots, \mu_r$ as above. In order that θ be nondegenerate, it is necessary and sufficient that for any $\alpha \in \mathfrak{A}^{-1}$, $x \in R$, the equations*

$$(*)\qquad \begin{aligned} x_{r-1}\mu_r + x_{r-2}\mu_{r-1} + \cdots + x_0\mu_1 &= \alpha, \\ x_{r-1}\mu_{r-1} + x_{r-2}u_{r-2} + \cdots + x_0\mu_0 &= x \end{aligned}$$

have simultaneous solutions with $x_j \in \mathfrak{A}^j$, $j = 0, \ldots, r-1$.

Proof. It follows directly from the definition that θ is nondegenerate iff given $\alpha \in \mathfrak{A}^{-1}$, $x \in R$ there are $a_1^i \cdots a_{r-1}^i \in \mathfrak{A}$, $z_1^i, \ldots, z_{r-1}^i \in R$, $i = 1, \ldots, m$, such that

$$\left(\sum_{i=1}^{m} a_1^i \cdots a_{r-1}^i\right)\mu_r + \left(\sum_{j=1}^{r-1}\sum_{i=1}^{m} a_1^i \cdots \hat{a}_j^i \cdots a_{r-1}^j\right) z_j^i\mu_{r-1}$$
$$+ \cdots + \left(\sum_{j_1<\cdots<j_t}\sum_{i=1}^{m} a_1^i \cdots \hat{a}_{j_1}^i \cdots \hat{a}_{j_2}^i \cdots \hat{a}_{j_t}^i \cdots a_{r-1}^i\right) z_{j_1}^i \cdots z_{j_t}^i \mu_{r-t}$$
$$+ \cdots + \left(\sum_{i=1}^{m} z_1^i \cdots z_{r-1}^i\right)\mu_1 = \alpha,$$

and

$$\left(\sum_{i=1}^{m} a_1^i \cdots a_{r-1}^i\right)\mu_{r-1} + \left(\sum_{j_1<\cdots<j_t}\sum_{i=1}^{m} a_1^i \cdots \hat{a}_{j_1}^i \cdots \hat{a}_{j_2}^i \cdots \hat{a}_{j_t}^i \cdots a_{r-1}^i\right)$$
$$z_{j_1}^i \cdots z_{j_t}^i \mu_{r-t-1} + \cdots + \left(\sum_{i=1}^{m} z_1^i \cdots z_{r-1}^i\right)\mu_0 = x.$$

(A caret over a symbol indicates its deletion.) Thus necessity is clear.

Now suppose we have a solution $x_0, \ldots, x_{r-1}$ to $(*)$, with $x_j \in \mathfrak{A}^j$, $j = 0, \ldots, r-1$. We show by induction that there are positive integers $M_1 < M_2 < \cdots < M_{r+1}$, such that for each $t \leq r$ and each $s \leq t$,

$$\sum_{j_1<\cdots<j_s}\sum_{i=1}^{M_{t+1}} a_1^i \cdots \hat{a}_{j_1}^i \cdots \hat{a}_{j_2}^i \cdots \hat{a}_{j_s}^i \cdots a_{r-1}^i z_{j_1}^i \cdots z_{j_s}^i = x_{r-s-1}.$$

Choose $a_1^i, \ldots, a_{r-1}^i \in \mathfrak{A}$, $i = 1, \ldots, M_1$, such that $\sum_{i=1}^{M_1} a_1^i \cdots a_{r-1}^i = x_{r-1}$. Choose $z_j^i = 0$ for $1 \leq j \leq r-1$, $i = 1, \ldots, M_1$.

Now suppose $t \geq 1$ and $M_1 < M_2 < \cdots < M_t$ have been chosen along with $a_1^i, \ldots, a_{r-1}^i$; $z_1^i, \ldots, z_{r-1}^i$ such that $\sum_{i=1}^{M_t} a_1^i \cdots a_{r-1}^i = x_{r-1}$ and

$$\sum_{1\leq k_1<\cdots<k_{j-1}\leq r-1}\sum_{i=1}^{M_t} a_1^i \cdots a_{k_1}^i \cdots \hat{a}_{k_2}^i \cdots \hat{a}_{k_{j-1}}^i z_{k_1}^i \cdots z_{k_{j-1}}^i = x_{r-j}$$

for $2 \leq j \leq t$.

Choose $M_{t+1} > M_t$ and for $i = M_t + 1, \ldots, M_{t+1}$, choose $a^i_{t+1}, \ldots, a^i_{r-1} \in \mathfrak{A}$ such that

$$\sum_{i=M_t+1}^{M_{t+1}} a^i_{j+1} \cdots a^i_{r-1} = x_{r-t-1}.$$

Let $a^i_1 = \cdots = a^i_t = 0$, $z^i_1 = \cdots = z^i_t = 1$, $z^i_{t+1} = \cdots = z^i_{r-1} = 0$ for $M_t < i \leq M_{t+1}$. Then by induction,

$$\sum_{i=1}^{M_{t+1}} a^i_1 \cdots a^i_{r-1} = \sum_{i=1}^{M_t} a^i_1 \cdots a^i_{t-1} = x_{r-1}$$

and

$$\sum_{j_1<\cdots<j_s\leq r-1} \sum_{i=1}^{M_{t+1}} a^i_1 \cdots \hat{a}^i_{j_1} \cdots \hat{a}^i_{j_2} \cdots \hat{a}^i_{j_s} \cdots a^i_{r-1} z^i_{j_1} \cdots z^i_{j_s}$$

$$= \sum_{j_1<\cdots<j_s} \sum_{i=1}^{M_t} a^i_1 \cdots \hat{a}^i_{j_1} \cdots \hat{a}^i_{j_2} \cdots \hat{a}^i_{j_s} \cdots a^i_{r-1} z^i_{j_1} \cdots z^i_{j_s}$$

$$= x_{r-s-1} \qquad \text{for } s < t.$$

Finally,

$$\sum_{j_1<\cdots<j_t} \sum_{i=1}^{M_{t+1}} a^i_1 \cdots \hat{a}^i_{j_1} \cdots \hat{a}^i_{j_2} \cdots \hat{a}^i_{j_t} \cdots a^i_{r-1} z^i_{j_1} \cdots z^i_{j_t}$$

$$= \sum_{i=M_t+1}^{M_{t+1}} a^i_{t+1} \cdots a^i_{r-1} = x_{r-t-1}.$$

Now the Lemma follows by induction, with $m = M_r$. □

We now suppose that $r \geq 3$, $\alpha \in \mathfrak{A}^{-1}$ and $x \in R$, and show there are $\mu_0, \ldots, \mu_r$ with $\mu_i \in \mathfrak{A}^{-i}$ such that (*) has a solution $x_1, \ldots, x_{r-1}$ with $x_j \in \mathfrak{A}^j$. To this end choose $\mu_0 = 1$, $\mu_1 = \cdots = \mu_{r-2} = 0$; and choose $\mu_{r-1} \in \mathfrak{A}^{1-r}$ and $\mu_r \in \mathfrak{A}^{-r}$ such that

$$\mu_r \mathfrak{A}^{r-1} + \mu_{r-1}\mathfrak{A}^{r-2} = \mathfrak{A}^{-1}.$$

We can do this since, in a Dedekind domain, any nonzero element of an ideal can be extended to a two element generating set for that ideal. Let $x_{r-1} \in \mathfrak{A}^{r-1}$, $x_{r-2} \in \mathfrak{A}^{r-2}$ be such that

$$\mu_r x_{r-1} + \mu_{r-1} x_{r-2} = \alpha.$$

Then let $x_0 = x - \mu_{r-1}x_{r-1}$. It is clear that $0, \ldots, 0, x_{r-2}, x_{r-1}$ is a solution to (*).

We summarize these results as follows.

Theorem. *Let R be a Dedekind domain and let V be a finitely generated projective R-module of* rank $n \geq 1$, *and let* $r \geq 2$. *Suppose* $V \cong \mathfrak{A} \oplus R \oplus \cdots \oplus R$ *where* $\mathfrak{A}$ *is a nonzero ideal in* R. *Then* V *admits a nondegenerate symmetric* r*-linear form iff*

(1) $r = 2$ *and* $\mathfrak{A}^2$ *is principal, or*

(2) $r \geq 3$, $n = 1$ *and* $\mathfrak{A}^r$ *is principal, or*

(3) $r \geq 3$, $n \geq 2$.

Note that $\mu_0, \ldots, \mu_r$ give a nondegenerate form on $\mathfrak{A} \oplus R$ iff the homomorphism from $\mathfrak{A}^{r-1} \oplus \mathfrak{A}^{r-2} \oplus \cdots \oplus \mathfrak{A} \oplus R$ to $\mathfrak{A}^{-1} \oplus R$ given by the matrix

$$\begin{pmatrix} \mu_r & \mu_{r-1} \cdots \mu_1 \\ \mu_{r-1} & \mu_{r-2} \cdots \mu_0 \end{pmatrix}$$

is surjective.

The referee has pointed out that the Theorem applies in case $R = D[x]$, where D is a Dedekind domain. See [2, Corollary (6.4)].

Finally, we answer a question raised by the referee, thanks to a conversation with Paul Eakin. Let R be a Noetherian domain of Krull dimension d. If $\mathfrak{A}$ is a projective ideal and if $r \geq d + 2$, then $\mathfrak{A} \oplus R$ has a nondegenerate symmetric form. To see this it suffices to produce $\mu_r, \mu_{r-1}, \ldots, \mu_{r-d}$, with $\mu_j \in \mathfrak{A}^{-j}$, such that $\mathfrak{A}^r\mu_r + \cdots + \mathfrak{A}^{r-d}\mu_{r-d} = R$. (For then these μ_j, along with $\mu_1 = 0$ will give a solution for the first equation in (*), with x_0 arbitrary. Then with $\mu_0 = 1$, we can choose x_0 so that the second equation holds.) Take $\mu_r = 1$ and choose μ_{r-1} so that $\mathfrak{A}^{r-1}\mu_{r-1}$ is not contained in any of the finite number of minimal primes over $\mathfrak{A}^r$. This is possible since $\mathfrak{A}^{r-1}$ is invertible. Then $\mathfrak{A}^r + \mathfrak{A}^{r-1}\mu_{r-1}$ has height $\leq d - 1$. If $\mathfrak{A}^r + \mathfrak{A}^{r-1}\mu_{r-1} \neq R$, choose μ_{r-2} so that $\mathfrak{A}^{r-2}\mu_{r-2}$ is not contained in any of the minimal primes over $\mathfrak{A}^r + \mathfrak{A}^{r-1}\mu_{r-1}$. Then $\mathfrak{A}^r + \mathfrak{A}^{r-1}\mu_{r-1} + \mathfrak{A}^{r-2}\mu_{r-2}$ has height $\leq d - 2$. Continuing in this manner, we get the desired $\mu_r, \ldots, \mu_{r-d}$.

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