A ONE DIMENSIONAL MANIFOLD IS OF
COHOMOLOGICAL DIMENSION 2

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ABSTRACT. G. Bredon defines the cohomological Dimension of a topological space X to be the supremum of all cohomological ϕ-dimensions of X, where ϕ varies over the entire families of supports on X. He has proved that if X is a topological n-manifold then the cohomological Dimension of X is n or $n + 1$. He was not able to decide which one it is, even for a space as simple as the real line. The objective of this paper is to solve his problem for $n = 1$. In particular, we have shown that the cohomological Dimension of the real line is 2.

Let ϕ be a family of supports on a topological space X. Godement [2] defines the ϕ-*dimension* ($\dim_\phi(X)$) of X to be the largest integer n (or ∞) if there is a sheaf $\mathcal{A}$ of abelian groups on X for which the Grothendieck cohomology [3] group $H_\phi^n(X, \mathcal{A}) \neq 0$. If ϕ runs over all those paracompactifying families of supports on X whose extents equal X, then $\dim_\phi(X)$ is independent of ϕ and is called the *cohomological dimension* ($\dim(X)$) of X. The class of all such spaces for which $\dim(X)$ has a meaning is large enough to include all locally paracompact Hausdorff spaces. The $\dim(X)$ of a n-manifold X is n and is dominated by the covering dimension [4] for paracompact Hausdorff X. Quite naturally, Bredon [1] defines the *cohomological Dimension* ($\mathrm{Dim}(X)$) of X to be the $\mathrm{Sup}_\phi\{\dim_\phi(X)\}$ for all families of supports ϕ on X. Among other things, he proves that for a topological n-manifold X, $\mathrm{Dim}(X) = n$ or $n + 1$ leaving it open which one it is. The present objective is to indicate its pathological or nonpathological nature by proving the following

Theorem. *Let X be a one-dimensional topological manifold. Then* $\mathrm{Dim}(X)$ *is two.*

Recall that if $\mathcal{A}$ is a sheaf of abelian groups on a space X and $\mathcal{A}^s(U)$ denotes the group of all serrations of $\mathcal{A}$ on U, then the sheaf generated by the presheaf $U \to \mathcal{A}^s(U)/\mathcal{A}(U)$ on X is denoted by $\mathcal{Z}^1(X, \mathcal{A})$ and $\mathrm{Dim}(X) \leq 1$ if and only if $\mathcal{Z}^1(X, \mathcal{A})$ is flabby for every sheaf $\mathcal{A}$ on X [1, p. 110]. Consider $\mathbf{R}$ as imbedded in the one-dimensional manifold X. Then the proof of the theorem follows from

Lemma. Let $A = \{0\} \cup \{1/m \,|\, m \in \mathbf{N}\}$ be the subset of X and $U = X - A$. Then the sheaf $\mathcal{A} = \mathbf{Z}_U$ on X is such that $\mathcal{Z}^1(X, \mathcal{A})$ is not flabby.

Proof. Notice that any section of $\mathcal{A}$ on any subset P of X such that $P \cap A \neq \varnothing$, is a zero section. Consider the set

$$\{f_m : (1/(m + 2), \, 1/m) \rightarrow \mathcal{A} \,|\, m \in \mathbf{N}\}$$

of serrations of $\mathcal{A}$ defined by

$$f_m(x) = \begin{cases} m, & x \neq 1/(m + 1), \\ 0, & x = 1/(m + 1). \end{cases}$$

These serrations give rise to a well-defined section S of $\mathcal{Z}^1(X, \mathcal{A})$ on $(0, 1)$ because of the fact that $f_{m+1} - f_m$ is a section of $\mathcal{A}$ on their common domain $(1/(m + 2), 1/(m + 1))$. We claim that this section cannot be extended to any neighbourhood of the set $[0, 1)$ in X. For, suppose S^1 is an extension of S in a neighbourhood $(-\delta, 1)$ of $[0, 1)$ where $\delta > 0$, and let S^1 be represented by a serration g of $\mathcal{A}$ defined on some neighbourhood $(-\epsilon, \epsilon)$ of $0 \in X$ where $\delta > \epsilon > 0$. Since S and S^1 agree on $(0, \epsilon)$, $g - f_m$ must be a section of $\mathcal{A}$ for each m sufficiently large. But that means, by our remark at the outset, that there is a positive integer N such that $\forall m \geq N$, $g - f_m = 0$ on $(1/(m + 2)$, $1/m)$. This means f_{N+k} and f_{N+k+1} must agree for each integer $k \geq 0$ on their common doamins, i.e., $N = N + k$, $\forall k \geq 0$, a contradiction.

Remark. It seems to me that for a topological n-manifold X, $\text{Dim}(X) = n + 1$ is true for every n. However, the difficulty in extending the proof lies in our inability to recognise precisely the stalks and the topology of the sheaf $\mathcal{Z}^n(X, \mathcal{A})$, $n \geq 1$, and for any given sheaf $\mathcal{A}$ on X.

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