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ON MAXIMAL TORSION RADICALS. III

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ABSTRACT. Maximal torsion radicals of a strongly semiprime ring correspond to minimal prime ideals and can be used to characterize both strongly semiprime and semiprime Goldie rings.

Let R be an associative ring with identity and let R-Mod denote the category of unital left R-modules. A subfunctor σ of the identity 1 on R-Mod is called a torsion preradical if $\sigma(M') = M' \cap \sigma(M)$ for all R-submodules $M' \subseteq M$, and a torsion radical if, in addition, $\sigma(M/\sigma(M)) = 0$ for all R-modules M. A torsion radical σ is called maximal if it is proper ($\sigma \neq 1$) and maximal with respect to the relation $\leq$, where $\tau \leq \sigma$ if $\tau(M) \subseteq \sigma(M)$ for all R-modules M. For ${}_RM$ the largest torsion radical σ such that $\sigma(M) = 0$ will be denoted $\mathrm{rad}_{E(M)}$, where $E(M)$ is the injective envelope of M. It is a well-known result that for a commutative Noetherian ring R there is a one-to-one correspondence between maximal torsion radicals of R-Mod and minimal prime ideals of R. This correspondence was recently extended by the author [2] to rings with Krull dimension on either the left or right. This paper shows that the correspondence holds for semiprime rings which satisfy a very weak form of d.c.c. on left annihilators, which holds in particular for any semiprime left Goldie ring.

Following the terminology of Handelman and Lawrence [6], a ring R will be called left strongly prime if for each $0 \neq b \in R$ there exist $r_1, r_2, \ldots, r_n \in R$ such that $ar_ib = 0$ for all i implies $a = 0$, for any $a \in R$. Such rings are just the left absolutely torsion free rings introduced by Rubin [7], who characterized them as rings R such that R-Mod has a maximal torsion radical which contains all proper torsion preradicals. Viola-Prioli [8] showed that a ring is left strongly prime if and only if every nonzero left ideal is cofaithful. (A module ${}_RM$ is cofaithful if there exists an embedding $0 \to R \to M^n$ of R in a finite direct sum of copies of M, or equivalently, if there exists a finite subset of M whose left annihilator is zero.) The notion can be extended as in [4] by calling a ring left strongly semiprime if it is semiprime and every faithful left ideal is cofaithful, the latter part of the condition being a weak form of d.c.c. on left annihilators. An equivalent condition is that every essential left ideal is cofaithful [5].

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Lemma (1). *If every faithful left ideal of R is cofaithful, then every proper right annihilator ideal of R is contained in a maximal right annihilator ideal.*

Proof. An ideal A is called a right annihilator if $A \neq 0$ and $A = r(S) = \{r \in R | Sr = 0\}$ for some subset $S \subseteq R$. (The left annihilator $\ell(S)$ is defined similarly.) If A is a proper right annihilator ideal of R, let $\{A_\alpha\}_{\alpha \in I}$ be any chain of proper right annihilator ideals each of which contains A. If $\ell(\bigcup_{\alpha \in I} A_\alpha) = 0$, then $\bigcup_{\alpha \in I} A_\alpha$ is faithful and therefore cofaithful by assumption, so there exist $a_1, \ldots, a_n \in \bigcup_{\alpha \in I} A_\alpha$ such that $\ell(a_1, \ldots, a_n) = 0$. Since $\{A_\alpha\}_{\alpha \in I}$ is a chain, $a_1, \ldots, a_n \in A_\alpha$ for some $\alpha \in I$, and then $\ell(A_\alpha) = 0$ and $A_\alpha = R$, a contradiction. Since $\ell(\bigcup_{\alpha \in I} A_\alpha) \neq 0$, $r\ell(\bigcup_{\alpha \in I} A_\alpha)$ is a proper right annihilator ideal and is a least upper bound for the chain $\{A_\alpha\}_{\alpha \in I}$. By Zorn's lemma there is a maximal element in the set of proper right annihilator ideals containing A. □

Theorem (2). *If R is a left strongly semiprime ring, then every torsion radical of R-*Mod *is contained in a maximal torsion radical, and there is a one-to-one correspondence between maximal torsion radicals of R-*Mod *and minimal prime ideals of R.*

Proof. Let $K \neq R$ be a nonzero torsion ideal ($K = \sigma(R)$ for some torsion radical σ) of the left strongly semiprime ring R. Then K is not cofaithful since $R \subseteq K^n$ would imply $K = R$. Since R is strongly semiprime, K is not faithful and hence $\ell(K) \neq 0$, so $r\ell(K)$ is a proper right annihilator ideal containing K. By Lemma 1 there is a maximal right annihilator ideal P which contains K, with $\ell(P) \subseteq \ell(K)$. Since R is semiprime, $\ell(K) \cap K = 0$ and $r(P) = \ell(P)$, so that P is the annihilator of a submodule of R/K isomorphic to $r(P)$, and therefore $\mathrm{rad}_{E(R/P)} \geq \mathrm{rad}_{E(R/K)}$. Since any maximal annihilator is a minimal prime, this shows that for any torsion ideal K there is a minimal prime ideal P such that $\mathrm{rad}_{E(R/P)} \geq \mathrm{rad}_{E(R/K)}$. (If R is prime, then 0 is a minimal prime, and if not, then as above there is a maximal annihilator P such that $\mathrm{rad}_{E(R/P)} \geq \mathrm{rad}_{E(R)}$.) By [2, Theorem 4.1], this establishes the theorem. □

The proof of Theorem 2 shows that every maximal torsion radical μ is of the form $\mathrm{rad}_{E(R/P)}$ for some maximal right annihilator ideal P, and since R is semiprime, P is a left annihilator; the proof of [1, Theorem 2.2] shows that since P is a closed ideal of a semiprime ring, $\mathrm{Hom}_R(P, E(R/P)) = 0$ and therefore $P = \mu(R)$. As a consequence of Theorem 2, the minimal primes of R are just the maximal right annihilator ideals. Moreover, there are only finitely many minimal primes (by the following theorem) generalizing the well-known result in the special case of semiprime left Goldie rings.

Theorem (3). *The ring R is left strongly semiprime $\Leftrightarrow$ every proper tor-*

sion preradical of R-Mod *is contained in a maximal torsion radical and* R-Mod *has only finitely many maximal torsion radicals.*

Proof. ($\Longrightarrow$) If τ is any proper torsion preradical, then $\tau(R)$ is not cofaithful, so as in the proof of Theorem 2 there is a maximal right annihilator ideal P with $\imath(P) \cap \tau(R) = 0$. Then $\tau(R/P) = 0$ since $\tau(\imath(P)) = 0$ and P can be embedded in a direct product of copies of $\imath(P)$. This implies that $\tau \leq \text{rad}_{E(R/P)}$, so every proper torsion preradical is contained in a maximal torsion radical. If $A_i \neq A_j$ are minimal left annihilator ideals of R, then $A_i \cap A_j = \ell(\imath(A_i) \cup \imath(A_j))$, which forces $A_i \cap A_j = 0$ and thus $A_i A_j = 0$ since R is semiprime. From this it follows that the sum in R of all minimal left annihilator ideals must be a direct sum, and then since R is left strongly semiprime, it cannot contain an infinite direct sum of ideals [4, Proposition 1.3], so the sum is finite. Thus there are only finitely many maximal right annihilator ideals $P_1, \ldots, P_n$ which determine maximal torsion radicals $\mu_1, \ldots, \mu_n$ with $\mu_i(R) = P_i$ for $i = 1, \ldots, n$.

($\Longleftarrow$) Let μ be any maximal torsion radical of R-Mod, with $\mu(R) = P$, let A/P be any nonzero left ideal of R/P, and let τ be the smallest torsion preradical such that $\tau(A/P) = A/P$ and $\tau \geq \mu$. Then $\tau = 1$ since $\tau \neq \mu$ and by assumption any proper torsion preradical must be contained in a maximal torsion radical. Since $\mu(E(R/P)) = 0$, $\tau(E(R/P))$ must be the sum of all homomorphic images of A/P, and so A/P generates $E(R/P)$, which implies that A/P is cofaithful as an R/P-module. Thus R/P is strongly prime since every nonzero left ideal is cofaithful.

Let $K = \bigcap_{i=1}^{n} \mu_i(R)$, where the intersection runs over all maximal torsion radicals, and let σ be the smallest torsion preradical such that $\sigma(R/K) = R/K$. Then $\sigma(R/\mu_i(R)) \neq 0$ since $R/\mu_i(R)$ is a homomorphic image of R/K, which implies $\sigma \not\leq \mu_i$ for all i, and therefore $\sigma = 1$. It follows that R/K is cofaithful, and so $K = 0$ and R is a finite subdirect product of strongly prime rings, which shows by [5, Proposition 5] that R is a strongly semiprime ring. □

The final theorem, when combined with Theorem 3, gives a characterization of semiprime Goldie rings which is stated in purely torsion theoretic language. Recall that a torsion radical σ is called perfect if the quotient functor defined by σ is naturally isomorphic to $Q_\sigma(R) \otimes_R -$, where $Q_\sigma(R)$ is the corresponding ring of quotients. If the ring $R/\sigma(R)$ has zero singular ideal, then the following are equivalent: (i) σ is perfect; (ii) $R/\sigma(R)$ has finite uniform dimension; (iii) $Q_\sigma(R)$ is semisimple Artinian [3, Proposition 3.9]. Furthermore, a semiprime ring is Goldie if and only if it has finite uniform dimension and zero singular ideal.

Theorem (4). *R is a semiprime left Goldie ring* $\Longleftrightarrow$ *R is left strongly* *semiprime and every maximal torsion radical of* R-Mod *is perfect.*

Proof. It can be shown that if R is strongly semiprime, then with the notation of Theorem 3, R is an essential R-submodule of the direct product $\prod_{i=1}^{n} R/P_i$, just as in [5, Proposition 9], and so $E(R) \simeq \bigoplus_{i=1}^{n} E(R/P_i)$. Since P_i is a torsion ideal, the R-injective envelope and R/P_i-injective envelope coincide, and thus $Q_{\max}(R) \simeq \prod_{i=1}^{n} Q_{\max}(R/P_i)$. (The complete ring of quotients $Q_{\max}(R)$ coincides with $E(R)$ since R has zero singular ideal, as shown by [5, Proposition 6]; the same is true of R/P_i.) From the remarks preceding the theorem, it follows immediately that R is Goldie if and only if $Q_{\max}(R)$ is semisimple Artinian, and this occurs if and only if each maximal torsion radical of R-Mod is perfect. □

REFERENCES

1. John A. Beachy, *On maximal torsion radicals*, Canad. J. Math. 25 (1973), 712–726.

2. ———, *On maximal torsion radicals*. II, Canad. J. Math. 27 (1975), 115–120.

3. ———, *Perfect quotient functors*, Comm. Algebra 2 (1974), 403–427.

4. John A. Beachy and William D. Blair, *Rings whose faithful left ideals are cofaithful*, Pacific J. Math. (to appear).

5. D. Handelman, *Strongly semiprime rings*, Pacific J. Math. (to appear).

6. D. Handelman and J. Lawrence, *Strongly prime rings*, Trans. Amer. Math. Soc. **211** (1975), 209–223.

7. Robert A. Rubin, *Absolutely torsion-free rings*, Pacific J. Math. 46 (1973), 503–514.

8. Jorge Viola-Prioli, *On absolutely torsion free rings and kernel functors*, Thesis, New Brunswick, N. J., Rutgers University, 1973.

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