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RESTRICTIONS OF ANALYTIC FUNCTIONS. III

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ABSTRACT. Characterizations are given of all real and all pure imaginary functions on a real Borel set which occur as restrictions of boundary functions of functions holomorphic and having positive imaginary part in the upper half-plane.

1. **Introduction.** Let f be a function which is holomorphic and has positive imaginary part in the upper half-plane Π_+. Then f is determined by the restriction of its boundary function to any real Borel set Δ of positive measure. Previous work [3], [4] has been concerned with characterizations of restrictions of this kind and a method by which the holomorphic function can be recaptured from a given restriction. To a large degree our work has been motivated by attempts to solve two problems concerning Hilbert transforms which arose out of [2]. Recall that if $f \in L^2(\Delta)$, then the Hilbert transform $\tau_\Delta f$ of f is defined a.e. on $(-\infty, \infty)$ by

$$(\tau_\Delta f)(x) = PV\frac{1}{\pi}\int_\Delta \frac{f(t)}{t-x}\,dt.$$

Problem I (*Generalized Loewner Problem*). Characterize all real valued measurable functions u on Δ such that

$$\frac{1}{\pi}\iint_{\Delta\times\Delta}{}' \frac{u(x)-u(t)}{x-t} f(t)f(x)^*\,dt\,dx \geq 0 \tag{1}$$

for all $f \in L^2(\Delta)$ such that $uf \in L^2(\Delta)$.

The integral in (1) is taken as

$$\lim_{\epsilon\searrow 0}\frac{1}{\pi}\iint_{\Delta\times\Delta}\chi_{\{|x-t|>\epsilon\}}(x,t)\,\frac{u(x)-u(t)}{x-t} f(t)f(x)^*\,dt\,dx$$
$$= \langle \tau_\Delta(uf), f\rangle_{L^2(\Delta)} - \langle \tau_\Delta f, uf\rangle_{L^2(\Delta)}.$$

Problem II. Characterize all nonnegative measurable functions v on Δ such that

$$\int_\Delta |(\tau_\Delta f)(t)|^2 v(t)\,dt \leq \int_\Delta |f(t)|^2 v(t)\,dt \tag{2}$$

for all $f \in L^2(\Delta)$ such that $vf \in L^2(\Delta)$.

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Quite remarkably, these problems have simple solutions involving functions in the class R of functions holomorphic and having nonnegative imaginary part in Π_+. In [3] we showed that a real valued function u on Δ is a solution to the generalized Loewner problem if and only if there exists w in R such that

$$u(x) = w(x + i0) \quad \text{a.e. on } \Delta. \tag{3}$$

In this paper we show that a nonnegative function v on Δ is a solution to Problem II if and only if there exists w in R such that

$$iv(x) = w(x + i0) \quad \text{a.e. on } \Delta. \tag{4}$$

2. **Exponential representations.** Let Δ be a real Borel set such that neither Δ nor its complement Δ^c is a Lebesgue null set. Let $U(\Delta)$ be the class of real valued functions u on Δ of form (3) for some $w \in R$, and let $V(\Delta)$ be the class of nonnegative functions v on Δ of form (4) for some $w \in R$.

In this section we collect three elementary results concerning exponential representations for functions in $U(\Delta)$ and $V(\Delta)$. From [1] we know that $w \in R$ if and only if w has a representation of the form

$$w(z) = C \exp\left(\int_{-\infty}^{+\infty} \frac{1+tz}{t-z}\,\frac{f(t)}{1+t^2}\,dt\right), \qquad z \in \Pi_+, \tag{5}$$

where C is a nonnegative constant and f is a measurable function such that $0 \le f \le 1$ a.e. on $(-\infty, \infty)$. Then

$$u(x + i0) = C \exp\left(PV\int_{-\infty}^{+\infty} \frac{1+tx}{t-x}\,\frac{f(t)}{1+t^2}\,dt + \pi i f(x)\right) \tag{6}$$

for almost all x in $(-\infty, \infty)$.

From (5) and (6) we easily derive the following two theorems.

Theorem 1. *Suppose $w \in R$.*

(i) $\operatorname{Im} w(x + i0) = 0$ *a.e. on Δ if and only if (5) holds, where $C \ge 0$, $0 \le f(t) \le 1$ a.e. on Δ^c, and $f(t) = \chi_E$ a.e. on Δ for some Borel subset E of Δ.*

(ii) $\operatorname{Re} w(x + i0) = 0$ *a.e. on Δ if and only if (5) holds, where $C \ge 0$, $0 \le f(t) \le 1$ a.e. on Δ^c, and $f(t) = ½$ a.e. on Δ.*

Theorem 2. (i) *A real valued function u on Δ is in $U(\Delta)$ if and only if u is given a.e. on Δ by*

$$u(x) = C\exp\left(PV\int_E \frac{1+tx}{t-x}\,\frac{dt}{1+t^2} + i\pi\chi_E(x)\right)\exp\left(PV\int_{\Delta^c} \frac{1+tx}{t-x}\,\frac{f(t)}{1+t^2}\,dt\right)$$

where $C \ge 0$, $0 \le f(t) \le 1$ a.e. on Δ^c, and E is a Borel subset of Δ.

(ii) *A nonnegative function v on Δ belongs to $V(\Delta)$ if and only if v is* *given a.e. on* Δ *by*

$$v(x) = C \exp\left(PV \frac{1}{2} \int_\Delta \frac{1+tx}{t-x} \frac{dt}{1+t^2}\right) \exp\left(PV \int_{\Delta^c} \frac{1+tx}{t-x} \frac{f(t)}{1+t^2}\, dt\right)$$

where $C \geq 0$ *and* $0 \leq f(t) \leq 1$ *a.e. on* Δ^c.

A bounded real valued function on $(-\infty, \infty)$ which occurs as the boundary function of an H^∞ function is necessarily a constant a.e. In contrast, there exist many bounded real valued functions on Δ which are restrictions of boundary functions of H^∞ functions.

Theorem 3. *A bounded real valued function* f *on* Δ *is the restriction of the boundary function of an* H^∞ *function if and only if* f *is given a.e. on* Δ *by*

$$f(x) = \lambda[v(x) - 1]/[v(x) + 1] \tag{7}$$

where λ *is a real constant and* $v \in V(\Delta)$.

Proof. This follows from the fact that $F \in H^\infty$ if and only if $F(z) = \lambda[W(z) - i]/[W(z) + i]$ where λ is a real constant and $W \in R$.

3. **Main result.** The following properties of τ_Δ will be useful for our consideration of Problem II.

(i) Suppose $f \in L^p(\Delta)$ where $p > 1$. Then $\tau_\Delta f \in L^p(\Delta)$ and τ_Δ is a bounded operator on $L^p(\Delta)$ to $L^p(\Delta)$.

(ii) Suppose $f, g \in L^2(\Delta)$. Then

$$\int_\Delta [(\tau_\Delta f)(t) g(t)^* + f(t)(\tau_\Delta g)(t)^*]\, dt = 0. \tag{8}$$

(iii) Suppose $f \in L^p(\Delta)$, $g \in L^q(\Delta)$ where $p^{-1} + q^{-1} < 1$. Then

$$\tau_\Delta[f(\tau_\Delta g) + (\tau_\Delta f) g] = (\tau_\Delta f)(\tau_\Delta g) - fg \quad \text{a.e. on } \Delta. \tag{9}$$

See [5, Chapter V] and [6, Chapter IV]. We first prove our main result in a special case.

Lemma. *Let* v *be a nonnegative function in* $L^1(\Delta) \cap L^\infty(\Delta)$. *Then the following statements are equivalent*:

(i) *for any* $f \in L^2(\Delta) \cap L^\infty(\Delta)$,

$$\int_\Delta |(\tau_\Delta f)(t)|^2 v(t)\, dt \leq \int_\Delta |f(t)|^2 v(t)\, dt;$$

(ii) $-\tau_\Delta v \in U(\Delta)$;

(iii) $v \in V(\Delta)$.

Proof. By (9), (i) is equivalent to having $-\int_\Delta (\tau_\Delta[f(\tau_\Delta f)^* + (\tau_\Delta f) f^*]) v\, dt \geq 0$ for all $f \in L^2(\Delta) \cap L^\infty(\Delta)$, and by (8) this holds if and only if

$$\int_\Delta [f(\tau_\Delta f)^* + (\tau_\Delta f)f^*](\tau_\Delta v)\,dt \geq 0$$

for all $f \in L^2(\Delta) \cap L^\infty(\Delta)$. By [3, Theorem 8] this last condition is equivalent to (ii).

Assume that (ii) holds, so there exists w in R such that $-(\tau_\Delta v)(x) = w(x + i0)$ a.e. on Δ. Then

$$p(z) = \frac{1}{\pi}\int_\Delta \frac{v(t)}{t - z}\,dt + w(z), \qquad z \in \Pi_+,$$

defines a function p in R such that

$$p(x + i0) = (\tau_\Delta v)(x) + iv(x) + w(x + i0) = iv(x) \quad \text{a.e. on } \Delta,$$

and therefore (iii) holds.

Assume that (iii) holds, so there exists p in R such that $iv(x) = p(x + i0)$ a.e. on Δ. Set

$$w(z) = p(z) - \frac{1}{\pi}\int_\Delta \frac{v(t)}{t - z}\,dt, \qquad z \in \Pi_+. \tag{10}$$

Since v is essentially bounded on Δ, Im $w(z)$ is bounded below in Π_+ and hence $\exp(iw(z))$ is in H^∞. But

$$\operatorname{Im} w(x + i0) = \chi_{\Delta^c}(x) \operatorname{Im} p(x + i0) \geq 0$$

a.e. on $(-\infty, \infty)$, so the boundary function of $\exp(iw(z))$ is bounded by 1 a.e. on $(-\infty, \infty)$. Therefore $\exp(iw(z))$ is bounded by 1 in Π_+ and $w \in R$. By construction

$$w(x + i0) = p(x + i0) - (\tau_\Delta v)(x) - i\chi_\Delta(x)v(x) = -(\tau_\Delta v)(x) \quad \text{a.e. on } \Delta,$$

so (ii) holds and the proof is complete.

Theorem 4. *A nonnegative measurable function v on Δ satisfies*

$$\int_\Delta |(\tau_\Delta f)(t)|^2 v(t)\,dt \leq \int_\Delta |f(t)|^2 v(t)\,dt \tag{11}$$

for all f in $L^2(\Delta)$ such that $vf \in L^2(\Delta)$ if and only if $v \in V(\Delta)$.

Proof. Let v be a nonnegative measurable function on Δ, and let $\Delta_1 \subseteq \Delta_2 \subseteq \cdots$ be Borel subsets of Δ such that $\bigcup_1^\infty \Delta_n = \Delta$ and the restriction v_n of v to Δ_n is in $L^1(\Delta_n) \cap L^\infty(\Delta_n)$.

If $v \in V(\Delta)$, then $v_n \in V(\Delta_n)$ for every $n = 1, 2, \ldots$. By the Lemma,

$$\int_{\Delta_n} |(\tau_{\Delta_n} f)(t)|^2 v_n(t)\,dt \leq \int_{\Delta_n} |f(t)|^2 v_n(t)\,dt \tag{12}$$

for all $f \in L^2(\Delta_n) \cap L^\infty(\Delta_n)$. A routine approximation argument shows that (11) holds for all $f \in L^2(\Delta)$ such that $vf \in L^2(\Delta)$.

Conversely suppose (11) holds for all $f \in L^2(\Delta)$ such that $vf \in L^2(\Delta)$.

Then (12) holds for all $f \in L^2(\Delta_n) \cap L^\infty(\Delta_n)$, $n = 1, 2, \ldots$. By the Lemma $v_n \in V(\Delta_n)$ and so there exists $w_n \in R$ such that $iv_n(x) = w_n(x + i0)$ a.e. on Δ_n. Since functions in R are determined by their boundary functions on sets of positive measure, $w_1(z) = w_2(z) = \cdots$ in Π_+, and denoting this function by $w(z)$ we obtain $iv(x) = w(x + i0)$ a.e. on Λ. Thus $v \in V(\Delta)$.

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