PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

A NOTE ON FINITE GROUPS HAVING A FIXED-POINT-FREE AUTOMORPHISM

MARTIN R. PETTET

ABSTRACT. A fusion result of Glauberman has as a consequence the fact that a finite group admitting a fixed-point-free automorphism has a normal Sylow 2-subgroup (and in particular, is solvable) if all nontrivial fixed-point subgroups have odd order.

The aim of this note is to point out a simple consequence of a recent fusion result of Glauberman which has an application to the problem of finite groups admitting a fixed-point-free automorphism. The general conjecture is that such groups are solvable. (See, for example, [3] and [2, Chapter 10].) We prove the following

Theorem. *Let G be a finite group admitting a fixed-point-free automorphism σ such that the fixed-point subgroup of each nontrivial power of σ has odd order. Then G is 2-closed.*

Note that 2-closure implies solvability by the Feit-Thompson theorem.

The argument is essentially a generalization of one used in the proof of Theorem 4.1 of [3].

Let $Qd(p)$ be the semidirect product of the 2-dimensional vector space over $GF(p)$ with its special linear group, and write $F(p)$ for the normalizer in $Qd(p)$ of a Sylow p-subgroup. Corollary 1 of Glauberman's paper [1] then states that if p is an odd prime, P is a Sylow p-subgroup of G, and $F(p)$ is not involved in $N_G(Z(J(P)))$, then $Z(J(P))$ controls strong fusion in P with respect to G.

Lemma. *A finite group G is 2-closed if and only if $N_G(Z(J(P)))$ is 2-closed for every odd order Sylow subgroup P of G.*

Proof. The "only if" direction is trivial, so assume that $N_G(Z(J(P)))$ is 2-closed for every odd order Sylow subgroup P. A Frattini argument shows that for odd primes p, normalizers of p-subgroups are preserved under the natural map from G to $G/O_2(G)$, so induction allows us to assume $O_2(G) = 1$. By Baer's theorem [2, Theorem 3.8.2], if x is an involution in G, the dihedral group $\langle x, x^g \rangle$ is not a 2-group for some element $g \in G$, so con-

Received by the editors June 17, 1974.
AMS (MOS) subject classifications (1970). Primary 20D10; Secondary 20D45.
Key words and phrases. Fixed-point-free automorphism, 2-closed, $Z(J(P))$ controls, strong fusion in P.

jugation by x must invert a nontrivial p-element y for some odd prime p. But if P is a Sylow p-subgroup of G containing y, the hypothesis that $N_G(Z(J(P)))$ is 2-closed implies both that $F(p)$ is not involved in $N_G(Z(J(P)))$ and also that no element of $N_G(Z(J(P)))$ can invert y. These two conclusions are in contradiction with the result of Glauberman mentioned above.

Before proving the theorem, we recall that if a finite group G admits a fixed-point-free automorphism σ, then for each prime p, G contains a unique σ-invariant Sylow p-subgroup which in turn contains every σ-invariant p-subgroup of G.

Proof of Theorem. We argue by induction on the order of G. If $N_G(Z(J(P))) \ntrianglelefteq G$ for every odd order σ-invariant Sylow subgroup P of G, we may simply apply induction and the above Lemma. Otherwise, $O_p(G) \neq 1$ for some odd prime p so, if S is the σ-invariant Sylow 2-subgroup of G, induction applied to $G/O_p(G)$ yields that $SO_p(G) \triangleleft G$. But by hypothesis, the action of σ on $SO_p(G)$ is such that the fixed points of every nontrivial power of σ lie in $O_p(G)$, so [4, Lemma 1] implies that $SO_p(G) = S \times O_p(G)$. Thus, $S \triangleleft G$ as required.

REFERENCES

1. G. Glauberman, *A sufficient condition for p-stability*, Proc. London Math. Soc. (3) 25 (1972), 253–287. MR 47 #1938.

2. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968. MR 38 #229.

3. E. W. Ralston, *Solvability of finite groups admitting fixed-point-free automorphisms of order rs*, J. Algebra 23 (1972), 164–180. MR 46 #1902.

4. B. Scimemi, *Finite groups admitting a fixed-point-free automorphism*, J. Algebra 10 (1968), 125–133. MR 37 #5289.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706