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CLASSIFYING MAPS IN FIBERINGS OF HOMOGENEOUS BOUNDED DOMAINS

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ABSTRACT. A homogeneous bounded domain in C^n admits a structure of a fiber space whose base and fibers are homogeneous bounded domains. For such a fibering there exists a universal fiber space with the same fibers, which plays an analogous role as a universal bundle does in topology. A sufficient condition is given in order that the canonical map of the base of the fibering into the classifying domain is injective. Some applications of it are also given.

It is known by Pjateckiĭ-Šapiro [4] that a homogeneous bounded domain D_0 admits a structure of a holomorphic fiber space over a homogeneous bounded domain D_2 with another homogeneous bounded domain D_1 as a standard fiber in such a way that the group of fiber-preserving holomorphic automorphisms is transitive on D_0. Such a fibering is called a homogeneous fibering of D_0. The universal and the classifying domains for a given homogeneous fibering are constructed also in [4] (see also [2]). In this note we give a sufficient condition (P) in terms of the Lie algebra of the automorphism group of D_0 in order that the classifying map of D_2 into the classifying domain is injective. The condition (P) is satisfied for the case where D_0 is an irreducible bounded symmetric domain. We consider also under what conditions a given fibering $D_0 \to D_2$ is itself the universal fibering. Throughout this note we will employ basic terminologies and techniques in [4] and [2].

1. We recall notations in [4] and [2]. Let $\{\mathfrak{t}_0, j, \omega\}$ be a normal j-algebra. Then $\mathfrak{t}_0$ has a structure of a graded Lie algebra $\mathfrak{t}_0 = \mathfrak{t}^0 + \mathfrak{t}^1 + \mathfrak{t}^{1/2}$ such that $\mathfrak{t}^0 = j\mathfrak{t}^1$, $j\mathfrak{t}^{1/2} = \mathfrak{t}^{1/2}$ and there exists an element $r \in \mathfrak{t}^1$ satisfying $[ja, r] = a$ for all $a \in \mathfrak{t}^1$. This decomposition is called the *canonical decomposition* of $\mathfrak{t}$. In the sequel we put $\mathfrak{t}^1 = R$, $\mathfrak{t}^{1/2} = W$. Let $R = \sum_{1 \le i \le k \le p} R^{ik}$, $jR = \sum_{1 \le i \le k \le p} jR^{ik}$ and $W = \sum_{1 \le i \le p} W^i$ be the decompositions into root spaces, where R^{ii} is one dimensional. The element r can be uniquely written as $r = \sum_{1 \le i \le p} r_i$ $(r_i \in R^{ii})$. Let $\{\psi_1, \ldots, \psi_p\}$ be the dual base of the base $\{jr_1, \ldots, jr_p\}$ of $\sum_i jR^{ii}$. Then the subspaces R^{ik}, jR^{ik}, W^i are the root spaces corresponding to the roots $(\psi_i + \psi_k)/2$,

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$(\psi_i - \psi_k)/2$ and $\psi_i/2$ respectively. Every root of $\mathfrak{t}$ is one of the above ones. Note that the root spaces are the common eigenspaces of the operators $\operatorname{ad} jr_k$ $(1 \leq k \leq p)$.

Let $\mathfrak{t}_1$ be a j-ideal of $\mathfrak{t}_0$. Putting $h(x, y) = \omega([jx, y]) + i\omega([x, y])$ $(x, y \in \mathfrak{t}_0)$, h is a positive definite hermitian form, and the orthogonal complement $\mathfrak{t}_2$ of $\mathfrak{t}_1$ with respect to h is a j-subalgebra; we have the decomposition

$$\mathfrak{t}_0 = \mathfrak{t}_1 + \mathfrak{t}_2. \tag{1}$$

Let $\mathfrak{t}_i = jR_i + R_i + W_i$ $(i = 1, 2)$ be the canonical decomposition. It is known that $[jR_1 + R_1, \mathfrak{t}_2] = 0$ and $[\mathfrak{t}_2, W_1] \subset W_1$. Let $\mathfrak{g}$ be the universal j-algebra of $\mathfrak{t}_1$ in the sense of [2]. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{t}_1 + \mathfrak{s}, \tag{2}$$

where $\mathfrak{s}$ is the *classifying j-algebra of* $\mathfrak{t}_1$ which is not only a semisimple j-subalgebra of $\mathfrak{g}$, but a subalgebra of $\mathfrak{gl}(W_1)$. The adjoint representation λ of $\mathfrak{t}_2$ on W_1 is a j-homomorphism of $\mathfrak{t}_2$ into $\mathfrak{s}$, which is called the *classifying j-homomorphism.* λ is naturally extended to a j-homomorphism $\tilde{\lambda}$ of $\mathfrak{t}_0$ into $\mathfrak{g}$ in such a way that $\tilde{\lambda}|\mathfrak{t}_1$ is the identity.

Let D_0 be a homogeneous bounded domain, $D_0 \xrightarrow{D_1} D_2$ a homogeneous fibering, and $\mathfrak{t}_i$ $(i = 0, 1, 2)$ be the Iwasawa j-algebra of D_i. Then $\mathfrak{t}_i$ is normal. It is known [4] that $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) can be chosen to be a j-ideal (resp. j-subalgebra) of $\mathfrak{t}_0$ and that the decomposition (1) holds. Let D and $D(\mathfrak{s})$ be the homogeneous bounded domains corresponding to $\mathfrak{g}$ and $\mathfrak{s}$, which are called the *universal* and the *classifying domains* of D_1, respectively. D has a structure of a fiber space over $D(\mathfrak{s})$ with fibers isomorphic to D_1. We have the following commutative diagram [2]

$$\begin{tikzcd}
D_0 \arrow[r, "\tilde{\mu}"] \arrow[d, "\pi_0"] & D \arrow[d, "\pi"] \\
D_2 \arrow[r, "\mu"] & D(\mathfrak{s})
\end{tikzcd}$$

where π_0 and π are the natural projections, and μ and $\tilde{\mu}$ are the holomorphic maps induced by λ and $\tilde{\lambda}$ respectively. Note that $D(\mathfrak{s})$ is a bounded symmetric domain.

2. Again consider the decompositions of R, jR and W into root spaces as in §1. Then, from the arguments in [4, pp. 65, 70–71] and the uniqueness of canonical decompositions [1], it follows that

$$
(3) \quad
\begin{aligned}
R_1 &= \sum_{1\le i\le k\le p_1} R^{ik}, & W_1 &= \sum_{1\le i\le p_1} W^i + \sum_{1\le i\le p_1;\, p_1+1\le k\le p} (R^{ik} + jR^{ik}),\\
R_2 &= \sum_{p_1+1\le i\le k\le p} R^{ik}, & W_2 &= \sum_{p_1+1\le i\le p} W^i.
\end{aligned}
$$

In the following, the indices $\alpha, \beta, \gamma, \cdots$ run through $1, \ldots, p_1$, and μ_0, $\lambda, \mu, \nu, \cdots$ through $p_1 + 1, \ldots, p$.

Lemma 1. *Let $R' = \Sigma_{\alpha,\mu} R^{\alpha\mu}$ and $W' = \Sigma_\alpha W^\alpha$. Then, for $a, b \in R_2$, $u \in W_2$ we have*

$$
[ja, W'] = 0, \quad [ja, R'] \subset R', \quad [ja, jR'] \subset jR',
$$
$$
[b, jR'] \subset R', \quad [u, W'] \subset R', \quad [u, jR'] \subset W'.
$$

Proof. Let $a = \Sigma_{\mu,\nu} a_{\mu\nu}$, $a_{\mu\nu} \in R^{\mu\nu}$, and let $u \in W^\alpha$. If $\mu \neq \nu$, we have $[ja_{\mu\nu}, u] = 0$, since $(\psi_\mu - \psi_\nu)/2 + \psi_\alpha/2$ is never a root. On the other hand $[jr_\mu, u] = \psi_\alpha(jr_\mu)u/2 = 0$, which implies $[ja_{\mu\mu}, u] = 0$. Thus $[ja, W'] = 0$. The sum $(\psi_\mu - \psi_\mu)/2 + (\psi_\alpha + \psi_\lambda)/2$ is a root if and only if $\lambda = \nu$, or $\mu = \nu$. We have then $[ja_{\mu\nu}, R^{\alpha\lambda}] \subset R^{\alpha\mu}$ or $R^{\alpha\lambda}$, according as $\lambda = \nu$ or $\mu = \nu$. This proves $[ja, R'] \subset R'$. Other relations are analogously proved.

We now introduce the following condition (P);

$$
(\mathrm{P}) \quad
\begin{array}{l}
\textit{for the subspace } R' = \Sigma_{\alpha,\mu} R^{\alpha\mu} \textit{ there exists an index}\\
\alpha\ (1 \le \alpha \le p_1) \textit{ such that } R^{\alpha\mu} \neq (0) \textit{ for each } \mu\ (p_1 + 1 \le \mu \le p).
\end{array}
$$

Using its realization as a Siegel domain [5], [6], we can verify that (P) is satisfied for the Iwasawa j-algebra $\mathfrak{t}_0$ of an irreducible bounded symmetric domain and each nontrivial j-ideal $\mathfrak{t}_1$.

Lemma A [3]. *For $x \in jR^{ik}$ $(i < k)$ and $z \in U^k = W^k + \Sigma_{k<l}(jR^{kl} + R^{kl})$, we have*

$$
h([x, z], [x, z]) = \lambda h(x, x) h(z, z),
$$

where λ is a nonzero constant.

In Lemmas 2–4 we assume condition (P).

Lemma 2. *If $[ja, jR'] = 0$, $a \in R_2$, then $a = 0$.*

Proof. Put $ja = \Sigma_{\mu\le\nu} ja_{\mu\nu}$, $a_{\mu\nu} \in R^{\mu\nu}$. By (P) we can choose a non-zero element $jx_{\alpha\mu_0} \in jR^{\alpha\mu_0}$, μ_0 being an arbitrarily fixed index. From the hypothesis we have

$$0 = [ja, jx_{\alpha\mu_0}] = \sum_{\mu_0 \leq \nu} [ja_{\mu_0\nu}, jx_{\alpha\mu_0}]$$

$$= \sum_{\mu_0 < \nu} [ja_{\mu_0\nu}, jx_{\alpha\mu_0}] + [ja_{\mu_0\mu_0}, jx_{\alpha\mu_0}].$$

Hence we have $[ja_{\mu_0\nu}, jx_{\alpha\mu_0}] = 0$ $(\nu > \mu_0)$ and $[ja_{\mu_0\mu_0}, jx_{\alpha\mu_0}] = 0$. Since $ja_{\mu_0\nu} \in U^{\mu_0}$, we have, from Lemma A, $ja_{\mu_0\nu} = 0$. On the other hand

$$[jr_{\mu_0}, jx_{\alpha\mu_0}] = j[jr_{\mu_0}, x_{\alpha\mu_0}] + j[r_{\mu_0}, jx_{\alpha\mu_0}]$$

$$= j(\tfrac{1}{2}(\psi_\alpha + \psi_{\mu_0})(jr_{\mu_0}))x_{\alpha\mu_0} + j[r, jx_{\alpha\mu_0}] = -\tfrac{1}{2} jx_{\alpha\mu_0},$$

from which $[ja_{\mu_0\mu_0}, jx_{\alpha\mu_0}] = 0$ implies $a_{\mu_0\mu_0} = 0$. So we have $a_{\mu_0\nu} = 0$ $(\nu \geq \mu_0)$. μ_0 being arbitrary, we get $a = 0$.

Using Lemma A, we have analogously:

Lemma 3. *If* $[u, jR'] = 0$, $u \in W_2$, *then* $u = 0$.

Lemma 4. *If* $[b, jR'] = 0$, $b \in R_2$, *then* $b = 0$.

Proof. Let $b = \Sigma_{\mu \leq \nu} b_{\mu\nu}$, $b_{\mu\nu} \in R^{\mu\nu}$. By (P) we can choose a nonzero element $jx_{\alpha\mu_0} \in R^{\alpha\mu_0}$, μ_0 being arbitrarily fixed. Then, from the hypothesis,

$$0 = [b, jx_{\alpha\mu_0}] = \sum_{\mu_0 < \nu} [b_{\mu_0\nu}, jx_{\alpha\mu_0}] + \sum_{\mu < \mu_0} [b_{\mu\mu_0}, jx_{\alpha\mu_0}]$$

$$+ [b_{\mu_0\mu_0}, jx_{\alpha\mu_0}].$$

So we get $[b_{\mu_0\nu}, jx_{\alpha\mu_0}] = 0$ $(\nu > \mu_0)$ and $[b_{\mu_0\mu_0}, jx_{\alpha\mu_0}] = 0$. Using Lemma A and an argument similar to that in Lemma 2, we conclude $b_{\mu_0\nu} = 0$ $(\nu \geq \mu_0)$. μ_0 being arbitrary, $b = 0$.

Proposition 1. *Let* $\mathfrak{t}_0$ *be a normal j-algebra and* $\mathfrak{t}_1$ *a nontrivial j-ideal of* $\mathfrak{t}_0$ *and* $\mathfrak{t}_0 = \mathfrak{t}_1 + \mathfrak{t}_2$ *be the decomposition* (1). *Suppose condition* (P) *is satisfied for* $\mathfrak{t}_0$ *and* $\mathfrak{t}_1$. *Then the classifying j-homomorphism* λ *of* $\mathfrak{t}_2$ *into* $\mathfrak{L}$ *is injective.*

Proof. Take an element $x = ja + b + u \in \mathfrak{t}_2$ $(a, b \in R_2, u \in W_2)$, and suppose $[x, W_1] = 0$. Then, by (3) we have

$$[ja, W'] + [ja, R'] + [ja, jR'] + [b, jR'] + [u, W'] + [u, jR'] = 0.$$

Therefore, by Lemma 1 we have

$$[ja, jR'] = 0, \qquad [u, jR'] = 0, \qquad [ja, R'] + [b, jR'] + [u, W'] = 0.$$

In view of Lemmas 2–4, we get $x = 0$.

Theorem 1. *Let* $D_0 \xrightarrow{D_1} D_2$ *be a homogeneous fibering of a homogeneous bounded domain* D_0. *If condition* (P) *is satisfied for the Iwasawa j-algebra* $\mathfrak{t}_0$ *of* D_0, *then the classifying map* μ *of* D_2 *into* $D(\mathfrak{g})$ *is injective, in particular,* μ *is a holomorphic imbedding.*

Proof. We will first recall the definition of μ [2]. Let T_2 be the simply connected Lie group corresponding to $\mathfrak{t}_2$. $D(\mathfrak{g})$ is represented as the coset space S/K, where S is a connected Lie group whose Lie algebra is $\mathfrak{g}$ and K is the (compact) isotropy subgroup at a point in $D(\mathfrak{g})$. The homomorphism λ of $\mathfrak{t}_2$ into $\mathfrak{g}$ is extended to a homomorphism of T_2 into S, which is denoted again by λ. Since T_2 acts on D_2 simply transitively, $y \in D_2$ is uniquely written as $y = t \cdot o$, where $t \in T_2$ and o is a fixed point of D_2. Then μ is defined by $\mu(y) = \lambda(t)K$. Take two points $y, y' \in D_2$ and let $y = t \cdot o$, $y' = t' \cdot o$ $(t, t' \in T_2)$. Suppose $\mu(y) = \mu(y')$. Then $\lambda(t^{-1}t') \in K$. On the other hand $\lambda(T_2)$ is **R**-triangular [2] and K is compact, so $\lambda(t^{-1}t')$ is the identity. By Proposition 1, λ is injective, since $\lambda(T_2)$ is simply connected, and so $y = y'$. It is obvious that μ is a holomorphic imbedding.

As was remarked before, condition (P) holds when D_0 is irreducible symmetric. So we have

Corollary. *Let* D_0 *be an irreducible bounded symmetric domain. Then, for any nontrivial homogeneous fibering of* D_0, *the classifying map* μ *is a holomorphic imbedding.*

Theorem 2. *Under the assumptions of Theorem* 1, D_0 *is itself the universal domain of* D_1 *provided that* $\dim D_2 = \dim D(\mathfrak{g})$.

Proof. $\mu(D_2)$ is the $\lambda(T_2)$-orbit of the origin of $D(\mathfrak{g}) = S/K$. From the fact that μ is an imbedding (Theorem 1) and $\dim D_2 = \dim D(\mathfrak{g})$, it follows that $\mu(D_2)$ is open in $D(\mathfrak{g})$. Since $D(\mathfrak{g})$ is connected homogeneous Kählerian, $\mu(D_2)$ must coincide with $D(\mathfrak{g})$. So the **R**-triangular subalgebra $\lambda(t_2)$ is a maximal **R**-triangular subalgebra of $\mathfrak{g}$. Therefore the subalgebra $\mathfrak{t}_{\mathfrak{g}} = \mathfrak{t}_1 + \lambda(\mathfrak{t}_2)$ is seen to be an Iwasawa j-algebra of D [2]. So $\tilde{\lambda}$ is a j-isomorphism of $\mathfrak{t}_0$ onto $\mathfrak{t}_{\mathfrak{g}}$, which implies that D_0 is holomorphically isomorphic to D.

Remark. The following facts are verified by using Theorem 2 and [5]: A classical domain D_0 of type I is universal if and only if the fiber D_1 is not isomorphic to a unit open ball. A classical domain D_0 of type II or III is always universal, while that of type IV is never universal if $\dim D_0 > 3$.

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