

## ANALYTIC TOEPLITZ OPERATORS WITH AUTOMORPHIC SYMBOL

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ABSTRACT. Let  $R$  denote the annulus  $\{z: \frac{1}{2} < |z| < 1\}$  and let  $\pi$  be a holomorphic universal covering map from the unit disk onto  $R$ . It is shown that if  $\pi$  is a function of an inner function  $\omega$ , that is, if  $\pi(z) = \pi(\omega(z))$ , then  $\omega$  is a linear fractional transformation. However, the analytic Toeplitz operator  $T_\pi$  has nontrivial reducing subspaces. These facts answer in the negative a question raised by Nordgren [10]. Let  $\phi$  be the function  $\phi(z) = \pi(z) - \frac{3}{4}$  and let  $\phi = \chi F$  be the inner-outer factorization of  $\phi$ . An operator  $C$  is produced which commutes with  $T_\phi$  but does not commute with  $T_\chi$  nor with  $T_F$ . This answers in the negative a question raised by Deddens and Wong [7]. The functions  $\pi$  and  $\phi$  are both automorphic under the group of covering transformations for  $\pi$  and hence may be viewed as functions on the annulus  $R$ . This point of view is critical in these examples.

Let  $D$  denote the open unit disk, let  $H^2$  denote the Hardy space of functions  $f$  analytic on  $D$  with  $\int |f(re^{i\theta})|^2 d\theta$  bounded independent of  $r$ , let  $H^\infty$  be the space of bounded analytic functions on  $D$ , and for  $\phi$  in  $H^\infty$  let  $T_\phi$  denote the operator on  $H^2$  defined by  $T_\phi(f) = \phi f$ . The operator  $T_\phi$  is said to be an analytic Toeplitz operator. A function  $\omega$  in  $H^\infty$  is said to be inner if  $\lim_{r \rightarrow 1} |\omega(re^{i\theta})| = 1$  for almost every  $\theta$  on the unit circle. If  $\omega$  is inner and nonconstant, then the analytic Toeplitz operator  $T_\omega$  is a unilateral shift. Moreover, this unilateral shift  $T_\omega$  has multiplicity one if and only if  $\omega$  is a linear fractional transformation. If  $\omega$  is not a linear fractional transformation, then  $T_\omega$  is a shift of multiplicity greater than one and has therefore many nontrivial reducing subspaces. Moreover, if  $\phi$  in  $H^\infty$  is a function of  $\omega$ , that is, if  $\phi(z) = \psi(\omega(z))$  for some  $\psi$  in  $H^\infty$ , then any reducing subspace for  $T_\omega$  also reduces  $T_\phi$  [10, Theorem 2]. Thus, if  $\phi$  is a function of an inner function which is not a linear fractional transformation, then  $T_\phi$  has nontrivial reducing subspaces. In [10], E. Nordgren inquires about the converse.

*Question 1.* If the analytic Toeplitz operator  $T_\phi$  has nontrivial reducing subspaces, must there be an inner function  $\omega$  which is not a linear fractional transformation and a function  $\psi$  in  $H^\infty$  such that  $\phi(z) = \psi(\omega(z))$  for all  $z$  in  $D$ ?

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An equivalent formulation of this question is given by J. Ball in [6].

A reducing subspace for an operator  $T$  is determined by a projection in the commutant  $\{T\}'$  of  $T$ . Thus, the problem of determining the reducing subspaces for the analytic Toeplitz operator  $T_\phi$  can be generalized to the problem of determining the commutant of  $T_\phi$ . In this regard, the following question has been posed by Deddens and Wong [7] and further pursued by Baker, Deddens, and Ullman [5].

*Question 2.* If  $\phi$  in  $H^\infty$  has inner-outer factorization  $\phi = \chi F$ , must  $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$ ?

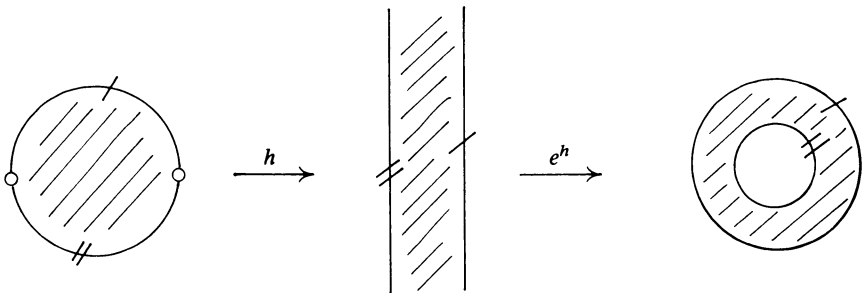
For a discussion of the inner-outer factorization of a function in  $H^\infty$ , see [9].

The purpose of this paper is to answer these two questions in the negative. The counterexamples are functions which are automorphic with respect to a certain group of linear fractional transformations, hence, the title of the paper. No attempt is made to systematically develop the theory of analytic Toeplitz operators with automorphic symbol.

The function in the first example is a holomorphic universal covering map  $\pi$  from the unit disk onto an annulus. Explicitly, let  $\text{Log}$  denote the principle branch of the logarithm and let

$$h(z) = \frac{i}{\pi} \log \frac{1}{2} \text{Log} \frac{1+z}{1-z} + \frac{1}{2} \log \frac{1}{2}.$$

The function  $h$  is a conformal map of the unit disk  $D$  onto the infinite strip  $\{z: \log \frac{1}{2} < \text{Re}(z) < 0\}$  which takes  $\{e^{i\theta}: 0 < \theta < \pi\}$  onto the imaginary axis and  $\{e^{i\theta}: \pi < \theta < 2\pi\}$  onto the line  $x = \log \frac{1}{2}$ . Define  $\pi(z) = e^{h(z)}$ . Then  $\pi$  is a locally conformal map from  $D$  onto the annulus  $R = \{z: \frac{1}{2} < |z| < 1\}$  which maps  $\{e^{i\theta}: 0 < \theta < \pi\}$  onto  $\{z: |z| = 1\}$  and maps  $\{e^{i\theta}: \pi < \theta < 2\pi\}$  onto  $\{z: |z| = \frac{1}{2}\}$ .



Further discussion of this function can be found in [11, p. 15].

**Theorem 1.** If  $\omega$  is an inner function and if  $\psi$  is a function in  $H^\infty$  with  $\pi(z) = \psi(\omega(z))$  for all  $z$  in  $D$ , then  $\omega$  is a linear fractional transformation.

**Proof.** The original proof of this theorem was long and cumbersome. The

following elegant proof was supplied by the reviewer. Assume  $\pi = \psi \circ \omega$ , where  $\omega$  is an analytic function mapping the unit disk onto itself which is not univalent. Let  $z_1$  and  $z_2$  be distinct points in the disk such that  $\omega(z_1) = \omega(z_2)$ . Then  $\pi(z_1) = \pi(z_2)$ , so  $z_1$  and  $z_2$  lie on a circular arc joining the points  $+1$  and  $-1$ . Let  $I$  be the closed subarc of that arc whose endpoints are  $z_1$  and  $z_2$ . For each  $z$  in  $I$ , the set  $I \cap \omega^{-1}(\omega(z))$  is finite. Moreover, because  $\omega$  is locally univalent (since  $\pi$  is), there is a positive lower bound for the distances between distinct points in any such set. Hence, we can find distinct points  $z_3$  and  $z_4$  on  $I$  such that  $\omega(z_3) = \omega(z_4)$ , but such that  $\omega(z) \neq \omega(z')$  whenever  $z$  and  $z'$  are distinct points of  $I$  satisfying  $|z - z'| < |z_3 - z_4|$ . Let  $J$  be the closed subarc of  $I$  with endpoints  $z_3$  and  $z_4$ . Then  $K = \omega(J)$  is a Jordan curve, and  $\psi(K)$  is a circle  $|z| = r$ , where  $\frac{1}{2} < r < 1$ . But then  $\psi$ , being an open map, must send the interior of  $K$  onto the disk  $|z| < r$ , which is impossible since  $\pi$  takes no values in the disk  $|z| < \frac{1}{2}$ .

A subnormal operator  $T$  is said to be pure (or completely nonnormal) if there are no nonzero reducing subspaces  $\mathfrak{M}$  such that  $T|_{\mathfrak{M}}$  is normal. Let  $\mu$  denote normalized linear Lebesgue measure on the unit circle and, for  $\phi$  in  $L^\infty(\mu)$ , let  $M_\phi$  denote the operator multiplication by  $\phi$  on  $L^2(\mu)$ . In what follows, a function  $\phi$  in  $H^\infty$  is identified with its boundary function in  $L^\infty(\mu)$ .

**Lemma.** *If  $\phi$  in  $H^\infty$  is nonconstant, then  $T_\phi$  is a pure subnormal operator and  $M_\phi$  is its minimal normal extension.*

**Proof.** These facts are proved by Douglas and Pearcy [8, Lemma 3.8].

**Lemma.** *The operator  $M_\pi$  is unitarily equivalent to multiplication by  $z$  on  $L^2_{\mathfrak{K}}(\partial R)$  where  $\mathfrak{K}$  is an infinite dimensional separable Hilbert space and the inner product is defined with respect to linear Lebesgue measure on the boundary of  $R$ .*

**Proof.** A study of multiplication operators by T. Kriete and the author shows that the scalar spectral measure of  $M_\pi$  is the measure  $\mu \circ \pi^{-1}$  and the multiplicity function for  $M_\pi$  is obtained by counting the number of points in the essential preimage [3]. The essential preimage under  $\pi$  of a point  $\lambda$  in  $\mathbb{C}$  is the set of points  $z$  in  $\partial D$  such that

$$\liminf_{\delta \rightarrow 0} \frac{\mu(V \cap \pi^{-1}(B_\delta(\lambda)))}{\mu(\pi^{-1}(B_\delta(\lambda)))}$$

is greater than zero for every open set  $V$  containing  $z$ . Here, the set  $B_\delta(\lambda)$  is the closed ball of radius  $\delta$  about  $\lambda$ .

Since  $\mu \circ \pi^{-1}$  is harmonic measure for the point  $\pi(0)$  [1, Lemma 1.11] which is absolutely continuous with respect to linear Lebesgue measure, the scalar spectral measure of  $M_\pi$  can be taken to be linear Lebesgue measure.

Furthermore, since the analytic function  $\pi$  is conformal in a neighborhood of  $\partial D \setminus \{-1, 1\}$ , it follows that for  $z$  in  $\partial D \setminus \{-1, 1\}$  and  $\pi(z) = \lambda$  (using the comment above on  $\mu \circ \pi^{-1}$  and omitting some routine computation)

$$\lim_{\delta \rightarrow 0} \frac{\mu(V \cap \pi^{-1}(B_\delta(\lambda)))}{\mu(\pi^{-1}(B_\delta(\lambda)))} = \frac{1}{|\pi'(z)|} > 0$$

for every open set  $V$  containing  $z$ . Hence, for all  $\lambda$  in  $\partial R$ , the essential preimage contains the preimage  $\pi^{-1}(\{\lambda\})$  which is infinite. Therefore, the multiplication operator  $M_\pi$  has uniform infinite multiplicity which completes the proof of the lemma.

**Theorem 2.** *The operator  $T_\pi$  has nontrivial reducing subspaces.*

**Proof.** By the first lemma, the operator  $T_\pi$  is a pure subnormal operator such that the spectrum of the minimal normal extension is contained in the boundary of  $R$ . Such operators are studied in general by R. G. Douglas and the author in [2]. There it is shown that such an operator is unitarily equivalent to a bundle shift over  $R$ , an operator which is multiplication by  $z$  on the  $H^2$ -space of cross-sections of a flat unitary vector bundle over  $R$  [2, Theorem 11]. It is also shown that any bundle shift over an annulus with multiplicity greater than one has nontrivial reducing subspaces [2, Theorem 7]. By the two lemmas above, the minimal normal extension of  $T_\pi$  has uniform infinite multiplicity and it follows that the unitarily equivalent bundle shift has infinite multiplicity [2, Theorem 3]. Hence, this bundle shift has nontrivial reducing subspaces which completes the proof of Theorem 2.

Counterexamples for the second question are obtained from certain general considerations. Let  $A$  be a linear fractional transformation mapping the unit disk  $D$  onto itself. A complex function  $f$  on  $D$  is said to be automorphic with respect to  $A$  if  $f(Az) = f(z)$  for all  $z$  in  $D$  and it is said to be modulus automorphic with respect to  $A$  if  $|f(Az)| = |f(z)|$  for all  $z$  in  $D$ . If  $f$  is analytic and modulus automorphic with respect to  $A$  and if  $f$  does not vanish identically, then  $(f \circ A)/f$  is meromorphic and unimodular. It follows that there is a constant  $\lambda$  of unit modulus such that  $f(Az) = \lambda f(z)$  for all  $z$  in  $D$ . The number  $\lambda$  is sometimes referred to as the index of  $f$ . Let  $C_A$  denote the composition operator on  $H^2$  defined by  $C_A(f)(z) = f(Az)$ .

**Lemma.** *If  $\phi$  in  $H^\infty$  is modulus automorphic with respect to  $A$  of index  $\lambda$ , then  $C_A T_\phi = \lambda T_\phi C_A$ . Hence, the operator  $C_A$  commutes with  $T_\phi$  if and only if  $\phi$  is automorphic.*

**Proof.** Evaluate.

**Lemma.** *If  $\phi$  in  $H^\infty$  is automorphic with respect to  $A$  and  $\phi = \chi F$  is*

the inner-outer factorization of  $\phi$ , then  $\chi$  and  $F$  are modulus automorphic with respect to  $A$ .

**Proof.** This lemma is proved by Voichick [12, Lemma 4.6].

By virtue of these two lemmas, to find a counterexample for Question 2 it is sufficient to produce a nontrivial linear fractional transformation  $A$  and a function  $\phi$  in  $H^\infty$  automorphic with respect to  $A$  such that its inner and outer factors are not automorphic with respect to  $A$ . It is well known that such functions exist; see, for example, [11]. A specific example can be written down in terms of the function  $\pi(z) = e^{h(z)}$  discussed above.

**Theorem 3.** *If  $A$  is the transformation  $A(z) = h^{-1}(h(z) + 2\pi i)$ , if  $\phi$  is the function  $\phi(z) = \pi(z) - \frac{3}{4}$ , and if  $\phi = \chi F$  is the inner-outer factorization of  $\phi$ , then  $A$  is a linear fractional transformation and  $C_A$  commutes with  $T_\phi$  but does not commute with  $T_\chi$  or  $T_F$ .*

**Proof.** Since  $A$  maps the unit disk conformally onto itself, it is a linear fractional transformation. Moreover,

$$\pi(Az) = e^{h(Az)} = e^{h(z)+2\pi i} = e^{h(z)} = \pi(z)$$

so that  $\pi$  and, hence,  $\phi$  are automorphic with respect to  $A$ . Now suppose that  $\chi$  is automorphic. Then  $\chi = k \circ \pi$  for some analytic function  $k$  on  $R$ . Since  $\pi$  is analytic across  $\partial D \setminus \{-1, 1\}$ , it follows that  $k$  is analytic across  $\partial R$ . Since  $|k(z)| = 1$  for  $z$  in  $\partial R$ , it follows from the argument principle that  $k$  has at least two zeros counting multiplicities in  $R$  [4, p. 7]. Let  $t$  in  $R$  be a zero of  $k$  and let  $\lambda$  in  $D$  be chosen so that  $t = \pi(\lambda)$ . Then

$$0 = 0F(\lambda) = k(t)F(\lambda) = k(\pi(\lambda))F(\lambda) = \phi(\lambda) = \pi(\lambda) - \frac{3}{4} = t - \frac{3}{4},$$

hence,  $t = \frac{3}{4}$ . Thus, the zero of  $k$  at  $t = \frac{3}{4}$  has multiplicity at least two, hence,  $k'(\frac{3}{4}) = 0$ . Therefore, if  $\lambda$  in  $D$  is chosen so that  $\pi(\lambda) = \frac{3}{4}$ , then

$$\begin{aligned} \pi'(\lambda) &= (\chi F)'(\lambda) = \chi(\lambda)F'(\lambda) + \chi'(\lambda)F(\lambda) = k(\pi(\lambda))F'(\lambda) + (k \circ \pi)'(\lambda)F(\lambda) \\ &= k(t)F'(\lambda) + k'(t)\pi'(\lambda)F(\lambda) = 0, \end{aligned}$$

a contradiction. This proves Theorem 3.

This completes the negative answers to Questions 1 and 2. It should be noted that the operator  $T_\pi$  also settles Questions 3 and 4 in [5], [7] in the negative. In fact, regarding Question 3, it follows from [2, Theorem 4] that the second commutant  $\{T_\pi\}''$  of  $T_\pi$  is algebraically isomorphic to  $H^\infty(R)$ , the Banach algebra of bounded analytic functions on  $R$ , while it is easily shown that for any inner function  $\omega$  the second commutant  $\{T_\omega\}''$  of  $T_\omega$  is algebraically isomorphic to  $H^\infty$ . Hence, the algebra  $\{T_\pi\}'$  is not even spatially isomorphic to  $\{T_\omega\}'$  for any inner function  $\omega$ .

In conclusion, a comment can be made on the source of these examples. Let  $G$  denote the group of linear fractional transformations  $\{A^n: n \text{ an integer}\}$  where  $A$  is defined in Theorem 3. The group  $G$  is said to be the group of covering transformations for the covering map  $\pi$ . A function  $f$  on  $D$  which is automorphic with respect to  $G$  can be written uniquely as  $f = g \circ \pi$  where  $g$  is a function on  $R$ . Informally, one says that a  $G$ -automorphic function on  $D$  can be viewed as a function on  $R$ . Thus, the  $G$ -automorphic function  $\pi$  viewed on  $R$  is just the function  $g(z) = z$  while the  $G$ -automorphic function  $\phi$  of Theorem 3 is the function  $g(z) = \frac{3}{4}z = z - \frac{3}{4}$ . Whereas these functions appear complicated as functions on  $D$ , they are simple when viewed as functions on  $R$ . The general strategy underlying the analysis in this paper is to make use of the alternate view of an automorphic function.

## BIBLIOGRAPHY

1. M. B. Abrahamse, *Toeplitz operators in multiply connected regions*, Bull. Amer. Math. Soc. 77 (1971), 449–454. MR 42 #8313.
2. M. B. Abrahamse and R. G. Douglas, *A class of subnormal operators related to multiply connected regions*, Advances in Math. (to appear).
3. M. B. Abrahamse and T. Kriete, *The spectral multiplicity of a multiplication operator*, Indiana J. Math. 22 (1973), 845–857.
4. L. V. Ahlfors, *Bounded analytic functions*, Duke Math. J. 14 (1947), 1–11. MR 9, 24.
5. I. N. Baker, J. A. Deddens and J. L. Ullman, *Entire Toeplitz operators* (to appear).
6. J. Ball, *Hardy space expectation operators and reducing subspaces* (preprint).
7. J. A. Deddens and T. K. Wong, *The commutant of analytic Toeplitz operators*, Trans. Amer. Math. Soc. 186 (1973), 261–273.
8. R. G. Douglas and C. Pearcy, *Spectral theory of generalized Toeplitz operators*, Trans. Amer. Math. Soc. 115 (1965), 433–444. MR 33 #7849.
9. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR 24 #A2844.
10. E. A. Nordgren, *Reducing subspaces of analytic Toeplitz operators*, Duke Math. J. 34 (1967), 175–181. MR 35 #7155.
11. D. Sarason, *The  $H^p$  spaces of an annulus*, Mem. Amer. Math. Soc. No. 56 (1965). MR 32 #6256.
12. M. Voichick, *Ideals and invariant subspaces of analytic functions*, Trans. Amer. Math. Soc. 111 (1964), 493–512. MR 28 #4129.

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