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MATRIX COMMUTATORS OVER AN ALGEBRAICALLY CLOSED FIELD

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ABSTRACT. Let A be an n-square matrix with zero trace over an algebraically closed field F, and let the characteristic of F not divide n. It is shown that A can be expressed as $A = XY - YX$ where the eigenvalues of X and Y may be arbitrarily specified as long as those of X are distinct.

Let F be a field, and let A be an n-square matrix over F. We say that A has property K if the following holds: If $\lambda_1, \lambda_2, \ldots, \lambda_{2n} \in F$ with $\lambda_i \neq \lambda_j$ when $1 \leq i < j \leq n$, then A can be written as a commutator $A = XY - YX$ where X and Y are matrices over F with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}$, respectively. Clearly if A has property K then $\mathrm{tr}(A) = 0$. C. R. Johnson [2] proved the converse in the case where F is the complex field. We show that every n-square nonscalar matrix is similar to a matrix with $n - 1$ zero diagonal entries, and then use this result and a theorem due to S. Friedland [1] to extend Johnson's result to arbitrary algebraically closed fields.

Let $A \dotplus B$ be the direct sum of the square matrices A and B, and let $A(i)$ be the principal submatrix of A that remains after row i and column i are removed. The set of all n-square matrices over F is denoted by $\Gamma_n(F)$.

Theorem 1. *If $A \in \Gamma_n(F)$ is not a scalar matrix then there exists $B = (b_{ij}) \in \Gamma_n(F)$ such that B is similar to A and $b_{ii} = 0$ for $i = 1, 2, \ldots, n - 1$.*

Proof. We induct on n. If $A \in \Gamma_n(F)$ is similar to the companion matrix of its characteristic polynomial, then the theorem holds. Hence, the theorem holds for $n = 2$. Suppose that $A \in \Gamma_3(F)$, A is not a scalar matrix, and A is not similar to the companion matrix of its characteristic polynomial. Then A is similar to a matrix of the form

$$C = \begin{bmatrix} 0 & a & 0 \\ 1 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

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where the polynomial $\lambda - c$ divides the polynomial $\lambda^2 - b\lambda - a$. If $b = c = 0$, then the theorem holds. If $b = c \neq 0$, then $a = 0$, and it is easy to see that C is similar to the matrix

$$\begin{bmatrix} 0 & -b & b \\ -b & 0 & b \\ -b & -b & 2b \end{bmatrix}.$$

Suppose that $b \neq c$, and let $D = b \dotplus c$. If we apply the theorem to D, we see that there exists a nonsingular $P \in \Gamma_2(F)$ such that if

$$B = (b_{ij}) = (1 \dotplus P)C(1 \dotplus P)^{-1}$$

then $b_{ii} = 0$ for $i = 1, 2$. Hence, the theorem holds for $n = 3$. Now suppose that the theorem holds for $n = m$ where $m \geq 3$. Let $A \in \Gamma_{m+1}(F)$ and not be a scalar matrix. Since A is not a scalar matrix we may assume that $A(m + 1)$ is not a scalar matrix. Applying the inductive assumption to $A(m + 1)$, we see that there exists a nonsingular $P \in \Gamma_m(F)$ such that if

$$C = (c_{ij}) = (P \dotplus 1)A(P \dotplus 1)^{-1}$$

then $c_{ii} = 0$ for $i = 1, 2, \ldots, m - 1$. If $c_{mm} = 0$ then the theorem follows. Suppose that $c_{mm} \neq 0$. Then $C(1)$ is not a scalar matrix. Therefore, applying the inductive assumption to $C(1)$, we see that there exists a nonsingular $Q \in \Gamma_m(F)$ such that if

$$B = (b_{ij}) = (1 \dotplus Q)C(1 \dotplus Q)^{-1}$$

then $b_{ii} = 0$ for $i = 1, 2, \ldots, m$. This proves the theorem.

Theorem 2. *Let F be an algebraically closed field and let* char$(F) \nmid n$. *If $A \in \Gamma_n(F)$ with* tr$(A) = 0$, *then A has property K.*

Proof. Clearly the theorem holds for $A = 0$. Suppose that $A \neq 0$. Since tr$(A) = 0$ and char$(F) \nmid n$, A is not a scalar matrix. Hence, by Theorem 1, since tr$(A) = 0$, there exists a nonsingular $P \in \Gamma_n(F)$ such that if $B = PAP^{-1}$ then $b_{ii} = 0$ for $i = 1, 2, \ldots, n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n} \in F$ with $\lambda_i \neq \lambda_j$ when $1 \leq i < j \leq n$. Let $U = \lambda_1 \dotplus \lambda_2 \dotplus \cdots \dotplus \lambda_n$, and let $V = (v_{ij}) \in \Gamma_n(F)$ such that

$$v_{ij} = b_{ij}/(\lambda_i - \lambda_j), \qquad i \neq j, \quad i, j = 1, 2, \ldots, n,$$

and $v_{11}, v_{22}, \ldots, v_{nn}$ are chosen [1] so that V has eigenvalues $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}$. Letting $X = P^{-1}UP$ and $Y = P^{-1}VP$, we see that $A = XY - YX$ where X and Y have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}$, respectively. Therefore, A has property K.

The requirement that F be algebraically closed cannot be removed unqualifiedly from Theorem 2. To see this, let $A \in \Gamma_2(F)$ such that A has no eigenvalues in F. Let $Y \in \Gamma_2(F)$ such that Y has two equal eigenvalues in F. Then Y is similar to a matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ where $a, b \in F$. Hence, if $X \in \Gamma_2(F)$, then the matrix $XY - YX$ has two eigenvalues in F. Therefore, A does not have property K.

The hypothesis that $\operatorname{char}(F) \nmid n$ in Theorem 2 can be replaced by the requirement that A not be a nonzero scalar matrix.

Theorem 3. *Let F be an algebraically closed field, and let $A \in \Gamma_n(F)$ with* $\operatorname{tr}(A) = 0$. *Then A has property K if and only if there does not exist a nonzero $a \in F$ such that $A = aI$.*

Proof. If A is not a nonzero scalar matrix, then a slight modification of the proof of Theorem 2 shows that A has property K. Suppose that $A = aI$ for some nonzero $a \in F$. Assume that A has property K. Then $A = XY - YX$ for some $X, Y \in \Gamma_n(F)$ where X has n distinct eigenvalues. Since A is a scalar matrix, we may assume that X is a diagonal matrix. However, if X is a diagonal matrix, then $XY - YX$ has all diagonal entries equal to zero. Since this contradicts $A = aI$ where $a \neq 0$, the theorem follows.

Added in proof. Theorem 1 of a paper by Joel Anderson and Joe Parker, Jr. [Linear and Multilinear Algebra 2 (1974), 203–209] which appeared after this note was accepted for publication implies our Theorem 2.

REFERENCES

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2. C. R. Johnson, *A note on matrix solutions to $A = XY - YX$*, Proc. Amer. Math. Soc. 42 (1974), 351–353.

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