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A NOTE ON TUTTE'S UNIMODULAR REPRESENTATION THEOREM

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ABSTRACT. A short, direct, and constructive proof is given of a result of W. T. Tutte (*Lectures on matroids*, J. Res. Nat. Bur. Standards Sect. B 69B (1965), 1–47).

Theorem. *Let G be an abstract finite combinatorial geometry whose dependencies can be represented by vectors over the field with two elements as well as by vectors over another field of characteristic other than two. Then G may be represented simultaneously over every field by the column vectors of a totally unimodular matrix.*

1. **Introduction.** A finite *combinatorial geometry* or *matroid* $F(S)$ is a finite set S of *points* along with a nonempty family of equicardinal subsets called *bases* which obey properties generalized from those of maximal independent sets of vectors in a vector space [1], [3]. Two geometries $G(S)$ and $H(T)$ are *isomorphic* if a bijection f takes points S of G onto points T of H in such a way that if $S' \subseteq S$, then S' is a basis of G if and only if $\{f(s) | s \in S'\}$ is a basis of H. Given a matrix M with entries in a field F, we may define a geometry whose points are column vectors of M and whose bases are maximal linearly independent sets of column vectors over F. Any geometry that arises from a matrix M in this way is called a *dependence geometry over* F and is again denoted by M. To avoid special arguments for trivial cases, we will assume in what follows that no matrix has a zero column. An arbitrary geometry G is *coordinatizable* over a field F if it is isomorphic to a dependence geometry M over F; in this case the matrix M is said to *represent* the geometry over F.

If M is a matrix with entries from a field F, then elementary row operations, deletion of zero rows in M, and multiplication of columns by nonzero scalars do not affect column independence. Thus matrices obtained from M by means of any of these operations produce dependence geometries which are isomorphic to the dependence geometry of M. We call geometry-preserving operations of the above three types *projective operations* and we say the resulting matrix $\bar{M}$ is *projectively equivalent* to M. If $|S| = n$ and if B is a basis of M of size r (where the rank of M is r), then we may assume

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that M is an echelon $r \times n$ matrix relative to the columns B (i.e., the columns B form an identity submatrix). By relabeling the columns of M if necessary, we may assume that the first r columns $\{c_1, \ldots, c_r\}$ form the identity matrix and that thus M is of the form $M = [IM']$. Then the following three facts are equivalent:

(1) the determinant of a submatrix of M' indexed by rows $\{r_{i_1}, \ldots, r_{i_k}\}$ and columns $\{c_{j_1}, \ldots, c_{j_k}\}$ does not vanish,

(2) the determinant of the $r \times r$ submatrix indexed by columns $B' = (B - \{c_{i_1}, \ldots, c_{i_k}\}) \cup \{c_{j_1}, \ldots, c_{j_k}\}$ does not vanish, and

(3) the columns B' form a basis.

Thus the geometric structure of M is determined by knowing precisely which square submatrices of M' are singular.

A *minor* of a geometry is the abstract analog of a submatrix; in fact, a geometry H is a minor of a dependence geometry M if and only if H is represented by the column vectors of a submatrix $\overline{M}'$ (without zero columns) of a matrix $\overline{M}$ where $\overline{M}$ is an echelon matrix projectively equivalent to M. It was an early result of Whitney that a geometry is coordinatizable over F_2, the field with two elements, if and only if it has no minor isomorphic to a geometry on four points all of whose two-element subsets are bases. This latter *obstruction* (to coordinatization over F_2) is the geometry represented over a field by a matrix of the form

$$L = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

where $abcd \neq 0$, $ad - bc \neq 0$. Then Tutte [3, Theorem 7.51] by using a number of powerful lemmas (including a representation theorem and a homotopy theorem) showed that a binary geometry G is representable over the rationals $\mathbf{Q}$ by a matrix M in which every subdeterminant (and entry) is 0, 1, or -1 if and only if G has no minor given by the following two matrices (over F_2):

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad P^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

(P represents the geometry associated with the projective space $PG(2, 2)$, while P^* represents the affine space $AG(3, 2)$ with a column deleted.) Such a matrix M is called *totally unimodular* and it serves to coordinatize G over every field simultaneously. A geometry which may be represented by a totally unimodular matrix is termed *unimodular* (*regular* in [3]). One can easily show that neither P nor P^* may be coordinatized over a field of characteristic other than two. Hence, Tutte's unimodular theorem yields as a corollary that a geometry G is unimodular if and only if it is coordinatizable over F_2 and some other field F of characteristic other than two.

The reason our proof of this corollary is considerably shorter and more constructive than Tutte's (and our result weaker) is that Tutte's starting point is the binary representing matrix M', while we start with the echelon matrix M representing G over F. We will show that a totally unimodular matrix $\overline{M}$ representing G can be constructed from M by projection operations (over F). (Tutte's method would involve a less natural assignment of -1's to some of the nonzero entries of M'.) We also remark that an unpublished result of R. Reid states that a geometry is coordinatizable over F_3 if and only if it does not have as a minor any of P, P^*, or the two geometries consisting of five points and all two (respectively three) element subsets as bases. Since these latter two geometries both contain L as a minor, a geometry which does not contain L, P, or P^* is representable over F_2 and F_3, so that Reid's result along with ours then yields Tutte's full unimodularity theorem. A proof of Reid's result will soon appear in a paper by R. Bixby.

2. The unimodularity theorem.

Theorem. *Let M be a matrix which represents G over a field F of characteristic different from two. Then G is binary if and only if we may perform projective operations on M to make it into a matrix $\overline{M}$ of zeros and ± 1's such that $\overline{M}$ represents G and is totally unimodular when viewed as a matrix over any field.*

Proof. Clearly a geometry represented by a totally unimodular matrix is binary. Conversely, assume we are given M. Then (by applying row operations and relabeling columns if necessary), we may assume M is an echelon matrix, $M = [IM']$.

The rest of the proof is in two parts. In (1) we invoke some of the results of [1] to give projective operations over F which transform M into a matrix $\overline{M}$ whose nonzero entries are all ± 1. In (2) we show that this matrix $\overline{M}$ is totally unimodular when considered over the rationals.

(1) Let $s = (f_1, \dots, f_k)$ be a maximum sequence of nonzero entries of M' such that one can find a sequence of lines (rows and columns) $(l_1, \dots, l_k)$ of M' such that after the lines $l_{i+1}, l_{i+2}, \dots, l_k$ have been deleted, f_i is the only entry in s in the line l_i $(i = 1, \dots, k)$. Then, by successively multiplying lines l_j by the entry $f_j'^{-1}$, we may set all the entries s equal to 1, where f_j' is the entry f_j after previous multiplications $(j = 1, 2, \dots, k)$. It was shown in [1, Proposition 2.7] that the size of s is $n - t$, where t is the number of components in a block decomposition of M'. It was further shown [1, Theorem 3.7] that every entry in M' (and we may assume M) will then be 0, 1, or -1. Briefly, the reason for this is that every nonzero entry is in one of the classes $A_0, A_1, \dots, A_m$, where $A_0 = s$ and an entry a in A_i is in a $p \times p$ submatrix M'' with exactly two nonzero entries in each line and with all its nonzero entries other than a in $A_0 \cup \dots \cup A_{i-1}$. Then by induction

all these latter entries of M'' are ± 1; since G is binary, $\det(M'') = 0$. But $\det(M'') = \pm 1 \pm a$ and this is zero over F if and only if $a = \mp 1$ (where the sign of a is determined by noting that there must be p (mod 2) -1's in M'').

(2) Now consider M as a matrix over the rationals. It is a $\{0, 1, -1\}$-matrix. We will show that whenever row operations over $\mathbf{Q}$ are performed on M so as to make a column c_j (with $c_{ij} = \pm 1$) into a column $\bar{c}_j$ with exactly one nonzero (± 1) entry $\bar{c}_{ij}$, the new matrix M remains a $\{0, 1, -1\}$-matrix over $\mathbf{Q}$ and continues to represent G over F. Every entry of c_j is 0, 1, or -1. Adding $-c_{ij}c_{hj}$ times row r_i to row r_h for all $h \neq i$, we get $\bar{c}_j$. Consider an entry c_{hk} for $h \neq i$, $k \neq j$. This entry becomes $\bar{c}_{hk} = c_{hk} - c_{ij}c_{hj}c_{ik}$. This is equal to 0, 1, or -1 unless $c_{hk}c_{ij}c_{hj}c_{ik} \neq 0$ and $c_{hk} = -c_{ij}c_{hj}c_{ik}$. But this cannot happen since then $c_{hk}c_{ij} - c_{hj}c_{ik} = \pm 2$ and G could not be binary (M would contain the minor L). Therefore $\bar{M}$ remains a $\{0, 1, -1\}$-matrix; since our row operation could just as well have been performed over F, $\bar{M}$ continues to represent G over F. Moreover, $\bar{M}$ is again an echelon matrix and the hypothesis of the Theorem is preserved.

Let S be any square submatrix of M. If a column of S is 0, $\det(S) = 0$ over $\mathbf{Q}$; otherwise for any column c_j such that $c_{ij} \neq 0$ and c_{ij} is in S, we may form $\bar{c}_j$ as in the preceding paragraph and preserve the $\mathbf{Q}$ determinant of S as well as the original hypothesis that M (and hence S) is an echelon $\{0, 1, -1\}$-matrix which represents a binary geometry over F. If we continue this process on entries of S, S eventually has a zero column, in which case $\det(S) = 0$ over $\mathbf{Q}$, or S becomes a monomial matrix (a $\{0, 1, -1\}$-matrix with exactly one nonzero entry in each line). But then $\det_{\mathbf{Q}}(S) = \pm 1$.

Note. This direct method of row reducing a matrix which does not contain L as a minor can be used to prove the necessity of Camion's condition for total unimodularity [2]: A $\{0, 1, -1\}$-matrix M over the rationals is totally unimodular if and only if $\sum_{c_{hk} \in S} c_{hk} = 0$ (mod 4) for every submatrix S of M with an even number of nonzero entries in each row and column.

Indeed, if M has a submatrix S with an even number of ± 1's in each row and column, $\sum c_{hk} = 2$ (mod 4), and c_{ij} is a nonzero element of S, then when $\bar{c}_j$ is formed as in (2) of the above proof, the cofactor of c_{ij} in S obeys the same conditions so that eventually L will appear as a minor and M cannot be totally unimodular.

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