ABSTRACT. In a previous paper the problem of constructing the integral closure of a finite integral domain \( k[x_1, \ldots, x_n] = \mathbb{k}[x] \) was considered. A reduction to the case \( \text{deg}(k(x)/k) = 1 \), \( k(x)/k \) separable, and \( n = 2 \) was made. A subsidiary problem was: if \( k[x] \) is not integrally closed, to find a \( y \) in \( k(x) \) integral over \( k[x] \) but not in it. This was done for \( n = 2 \), but should have been done for arbitrary \( n \). The extra details are here given. For the convenience of the reader, the full argument is sketched.

In [2] we proposed to construct the integral closure of a finite integral domain \( k[x_1, \ldots, x_n] = \mathbb{k}[x] \) in its quotient field \( k(x_1, \ldots, x_n) \). Three subsidiary problems were formulated, of which the first two were:

1. to give a method for deciding whether \( k[x] \) is integrally closed;
2. in the case \( k[x] \) is not integrally closed, to give a method for finding an element in \( k(x) \) integral over \( k[x] \) but not in it.

We dealt first with the case that \( k(x)/k \) is separable, and a reduction to the case degree of transcendency of \( k(x)/k = 1 \) was made. It is then easy to reduce the original problem to the case \( n = 2 \), but on p. 7 it was stated, though incorrectly, that the subsidiary problem 2 was thus reduced. The slip was (in effect) noted in [1]. This is a note of correction. Basically we assume a familiarity with [2], but, for the convenience of the reader, try to rely on [2] as little as possible.

For another treatment (not quite complete) of the problems here considered see [5].
A \cap B, A : B \ (S_{2}S_{3}, 3) and an integer p such that A : B^{p} = A : B^{p+1} (S_{20}).

These constructions hold for any explicitly given field k.

The ring \( k[x_{1},\ldots,x_{n}] = k[x] \) (whose integral closure \( k[x]^{*} \) is sought) is given as \( k[X] \mod \text{the ideal } P \) of relations satisfied by \( x/k \). Contracting \( A = P \) to \( k[X_{i_{1}},\ldots,X_{i_{s}}] \), one can test the algebraic independence of \( x_{i_{1}},\ldots,x_{i_{s}} \) over k, and in this way find a transcendency basis of \( k(x)/k \) amongst the \( x_{i_{1}} \); say this is \( (x_{1},\ldots,x_{r}) \). If \( y = f(x)/g(x) \) with \( f, g \in k[X] \), then \( \bigcup_{i \in P} y g(X) - f(X) : g(X)^{p} \) is, as one checks, the defining ideal of \( (x, y)/k \), and contracting this to \( k[X_{1},\ldots,X_{r},Y] \) one finds the defining equation of \( y \) over \( k(x_{1},\ldots,x_{r}) \). If \( y \) is integral over \( k[x_{1},\ldots,x_{r}] \), then (since this ring is integrally closed) the equation will be an equation of integral dependence. Hence, though we omit some details, for \( y \in k(x) \), we can test the integral dependence of \( y/k[x] \) and, if \( y \) is integral, construct an equation showing this.

In [2] (cf. also [3]) we introduced a condition (P) for explicitly given fields k that in effect allows us to check for \( p \)-independence in \( k \) (i.e., if \( a_{1},\ldots,a_{s} \in k \), whether \( [k^{p}(a_{1},\ldots,a_{s}): k^{p}] = p^{s} \)). Our problems are to be solved for \( k \) satisfying (P), a condition void for explicitly given \( k \) of characteristic 0. (For the role of (P) see [2] and reference 6 in [3].)

Let \( u \) be an indeterminate and \( K = k(u) \). If \( y_{1},\ldots,y_{m} \) are \( k(u)[x] \)-module generators for the integral closure \( k(u)[x]^{*} \) of \( k(u)[x] \), we may, multiplying the \( y_{i} \) by a denominator \( d(u) \in k[u] \), suppose the \( y_{i} \) integral over \( k[x,u] \), hence in \( k(x)[u] \). Writing the \( y_{i} \) as polynomials in \( u \), the coefficients are in \( k[x]^{*} \) (since \( k[x]^{*}[u] \) is integrally closed) and yield a \( k[x] \)-module basis of \( k[x]^{*} \). Thus in solving our main problem we may freely adjoin indeterminates to \( k \); in particular, we may assume \( k \) infinite. By [2, p. 9] a similar technique is available for \( K/k \) finite and \( K \) linearly disjoint from \( k(x)/k \), a result we use only for \( d k(x)/k = 1 \) and \( K = k(a^{1/p}) \) with \( a \in k \).

2. The construction. Let \( V \) be the variety having \( (x_{1},\ldots,x_{n}) \) as generic point over \( k \); \( k[x] \) is integrally closed if and only if \( V \) is normal. Let \( r = \dim V \); we may as well suppose \( r \geq 1 \). Using the mixed-Jacobian of Zariski (cf. [4, p. 360]) and (P), we can write down an ideal \( A \) in \( k[X] \) for the singularities of \( V/k \) and find its dimension. If \( V \) has a singularity of codimension 1, it is certainly not normal (cf. [4]). Assume \( V \) has no singularity of codimension 1. By [2, p. 10] or the reference to F. K. Schmidt in [4, p. 376], one can construct an element \( c \neq 0 \) in the conductor of \( k[x] \); for \( k(x)/k \) separable, see [4, p. 365]. If \( (c) = (1) \), which we can decide, then \( V \) is normal, and if \( (c) \neq (1) \), then by [2, p. 5] or [4, p. 363f], \( V \) is normal if and only if \( (c) \) is unmixed. Let \( (c) = Q_{1} \cap \cdots \cap Q_{s} \cap \cdots \cap Q_{t} \) be a normal decomposition of \( (c) \) into primary
ideals with $Q_1, \ldots, Q_s$ the primaries belonging to the minimal primes of $(c)$. We can construct $Q_1 \cap \cdots \cap Q_s$ and compare it with $(c)$; assume $(c)$ is mixed. Let $d \in Q_1 \cap \cdots \cap Q_s$ not in $(c)$. Taking note that the local rings of $k[x]$ with respect to its minimal primes are the same as the local rings of $k[x]^*$ with respect to its minimal primes, we see that $d/c$ is integral over $k[x]$ but not in it. The same reasoning shows that $Q_1 \cap \cdots \cap Q_s/c$ is the full integral closure of $k[x]$. Thus problem 1 is solved and so is the main problem if $V$ has no singularity of codimension 1.

Assuming $r > 1$, we cut $V$ by a generic hyperplane $H$. Let $u_1, \ldots, u_n$ be indeterminates and place $z = u_1 x_1 + \cdots + u_n x_n$. Then by [4, p. 367], the ring of this section is $k(u, z)[x]$. Now let $u_{i1}, \ldots, u_{in}, i = 1, \ldots, r - 1$, be indeterminates and place $z_i = u_{i1} x_1 + \cdots + u_{in} x_n, i = 1, \ldots, r - 1$. Then $k(u, z)[x]$ is 1-dimensional. Assuming the main problem solved for $r = 1$, let $y_1, \ldots, y_m$ be a $k(u, z)[x]$-module basis of $k(u, z)[x]^*$; we may suppose $y_1, \ldots, y_m$ are integral over $k(u)[z, x] = k(u, z)[x]$. Further, by an argument given above, we may suppose $y_1, \ldots, y_m$ to be in $k(x)$; they are, then, integral over $k[x]$. It is not to be expected that the variety $V'$ having $(x, y)$ as generic point over $k$ is normal; however, it is free of singularities of codimension 1. In fact, suppose $p$ is an $(r - 1)$-dimensional prime in $k[x, y]$ such that its local ring $k[x, y]_p$ is not regular. Then also the local ring $k(u)[x, y]_p$ is not regular; but (since the $z_i$ are algebraically independent over $k(u)$ mod $p$) this is the same as the local ring of $k(u, z)[x, y]_p$ in $k(u, z)[x, y]$, a contradiction. Hence, by the preceding paragraph, we can construct $k[x, y]^* = k[x]^*$. Thus the main problem is reduced to $r = 1$. (Cf. [2, p. 6f].)

Let, then, $(x_1, \ldots, x_n)$ be a generic point for a curve $V/k$. We (first) assume $k(x)/k$ separable. By our condition (P), which allows us to check for $p$-independence, we can decide whether a given element $a$ of $k$ has a $p$th root in $k$ ($p =$ characteristic); and if it does not, then adjoining $a^{1/p}$ to $k$ we get an extension of $k$ linearly disjoint from $k(x)/k$. As mentioned, if we can solve our (main) problem over $k(a^{1/p})$ we can work back to get a solution over $k$. Hence we can freely adjoin $p$th roots to our base field. We are given the ideal $P$ of relations of $(x_1, \ldots, x_n)/k$, via a basis, and we adjoin the $p$th roots of the coefficients of the basis elements to $k$ and may thus suppose they are in $k$. The result is that the singularities of $V$ (over the new $k$) become absolute (in effect, given by the Jacobian rather than the mixed-Jacobian of the basis). $V$ may lose its singularities and thus become normal in the process, but this makes no difference.

Assume for a moment that $k$ is algebraically closed. Let $P$ be a point on $V$, say the origin. If no branch of $V$ centered at $P$ has its tangent in $X_1 = 0$ (equivalently: if $X_1 = 0$ is not tangent to $V$ at $P$), then $s(x_1) \leq$
u(x_i) in every branch centered at P, i = 2, · · · , n, and hence x_i/x_1 is integral over the local ring at P. Even if V is tangent to X_1 = 0 at P, but assuming V is not in X_1 = 0, one can compute a ρ such that x_i^ρ/x_1 is integral over the local ring at P; in fact obviously ρ = order of V will do.

Let (x_1, · · · , x_n) be a generic point of V/k, k(x)/k separable, and, as above, with the singularities of V absolute. We subject V to a generic homogeneous nonsingular linear transformation. Here we adjoin n^2 indeterminates u_{ij} to k, but, as explained, we can later remove them, so we write k for k(u); and x_1, · · · , x_n for the "transformed" variables. As mentioned, we can compute a polynomial g_i(X) ∈ k[X] whose roots are precisely the ith coordinates of the singularities of V. Then, as in [2, p. 20], after extending k by some p-th roots, we get a polynomial g_i(X) ∈ k[X] having these coordinates as roots with multiplicity 1. Let V be the variety having (x_1, · · · , x_n) as generic point over k, the algebraic closure of k. Because of the nonspecial position of V, by an argument given above, g_i(x_i)/g_1(x_1) will be integral over the local ring of V/k of each point of V that is singular for V/k; it will also be in the local ring of any point P for which g_i(P) ≠ 0. There remain the simple points (of V) on g_i(X_1) = 0. Because of the generic direction of X_i = 0, no two points of V ∩ (g_i(X_1) = 0) have the same ith coordinates (i > 1). We can compute a polynomial P_i(X) having these ith coordinates as roots, and, as before, with multiplicity 1. Then P_i = P_i^*/g_i will be a polynomial over k having as roots the ith coordinates of the simple points (of V) on g_i(X_1) = 0; and G.C.D. (g_i, P_i) = 1. As above, we can compute a ρ such that z_i = g_i(x_i)P_i(x_i)/g_1(x_1) is integral over every local ring of V/k, hence integral over k[x], and over k[x_i]. If V has singularities, the z_i will not all be in k[x], for if P is one such singularity and z_i ∈ Q(P/V), the local ring, then from g_i(x_i)(P_i(x_i))^ρ/g_1(x_1) ∈ Q(P/V) we get a polynomial in k[X_1 - X_1(P), · · · , X_n - X_n(P)] vanishing over V and having for linear terms a linear term in X_i - X_i(P); if this happens for all i, then P is simple on V by the Jacobian criterion. Hence at least one z_i is not in k[x]; it is easy to decide which z_i are in k[x], after converting this question into one on polynomials. This solves problem 2 (for dim V = 1, k(x)/k separable, and for an augmented k).

After a nonspecial homogeneous linear transformation on (x_1, · · · , x_n) over k, we may suppose k[x] is integral over k[x_1]. Let w_1, · · · , w_m be a linear basis of k(x)/k(x_1). Place Tr w_i w_j = Σ w_i w_j(k), where the superscript indicates conjugation/k(x_1). Then d(w) = det(Tr w_i w_j) is the discriminant of the basis w_1, · · · , w_m and is ≠ 0. If w'_i = Σ a_{ij} w_j, with a_{ij} ∈ k(x_1), is another basis of k(x)/k(x_1), then d(w') = (det A^2)d(w); here A = ∥a_{ij}∥. Now let w_1, · · · , w_m be integral over k[x_1], whence d(w) ∈ k[x_1], and let w'_1 be integral over k[x_1, w_1, · · · , w_m] but not in it. Write w'_1 =
(a_1w_1 + \cdots + a_mw_m)/c, \text{ with } a_i, c \in \mathbb{k}[x]. \text{ We may assume } a_i = 0 \text{ or }
\deg a_i < \deg c, i = 1, \ldots, m; \text{ and at least one } a_i \neq 0, \text{ say } a_1 \neq 0. \text{ Place }

w'_2 = w_2, \ldots, w'_m = w_m. \text{ Then } d(w') = (a_1^2/c^2)d(w), \text{ whence } \deg d(w') < \deg d(w). \text{ Starting with } w_1, \ldots, w_m \text{ in } \mathbb{k}[x], \text{ the process can be applied at most } d(w) \text{ times, a bound that does not change even upon successive ad-
junctions to } k \text{ of indeterminates and } p\text{th roots. Hence we soon get to the inte-
gral closure of } \mathbb{k}[x] \text{ and the main problem is solved, over an augmented }
k. \text{ As mentioned earlier, we can work back to the original } k. \text{ Now the main }
problem (and with it problem 2) is solved for any explicitly given } k \text{ satis-
ifying (P) and } k(x)/k \text{ separable.}

The above construction does not use our condition (F), the condition
that one should be able to factor a polynomial effectively over } k. \text{ Cf. [2, pp. 8, 16].}

Finally, there is the problem (for } r = 1) \text{ of reducing to the separable
case; this is done on pp. 9–10 of [2] and involves successive ad-
njunctions of } p\text{th roots to the base field: any such extension is either inner, i.e., for }
a p\text{th root } a^{1/p}, a^{1/p} \in k(x), \text{ or outer, i.e., } a^{1/p} \notin k(x); \text{ and assuming con-
dition (P) for } k, \text{ we can decide which by } [2, \text{ p. } 12] \text{ or } [3, \text{ §40}.] \text{ An outer ex-
tension yields a field } k(a^{1/p}) \text{ linearly disjoint from } k(x)/k; \text{ and we have
said above how to meet this. If the extension is inner, the ring to be con-
structed does not change, but we have to compute the ideal of relations for
(x_1, \ldots, x_n) \text{ over the (new) base field } k(a^{1/p}); \text{ if } a^{1/p} = f(x)/g(x) \text{ with } f, 
g \in k[x], \text{ then this is } (P, a^{1/p}\cdot g(X) - f(X)): g(X)^{p-1}, \text{ as one easily checks.}

The third subsidiary problem was to count the number of steps. The
above considerations involve no new difficulty in this regard.

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