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MODULES SATISFYING BOTH CHAIN CONDITIONS WITH RESPECT TO A TORSION THEORY

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ABSTRACT. Goldman [3] has introduced the notion of the length of a module with respect to a torsion theory and has studied finitely-generated modules over left noetherian rings which have finite length. In this note we simplify the proofs of some of Goldman's results and generalize them by removing both the finite-generation and noetherianness conditions.

0. **Background and notation.** Throughout the following, R will denote an associative (but not necessarily commutative) ring with unit element 1. We will denote by R-mod the category of all unitary left R-modules and will abuse notation by writing $M \in R$-mod when we mean to say that M is an object of R-mod.

The set of all (hereditary) torsion theories on R-mod will be denoted by R-tors. The reader is referred to [2], [4], [7] for basic information about such theories. If $\tau \in R$-tors we denote the class of all τ-torsion left R-modules by $\mathcal{T}_\tau$ and the class of all τ-torsion-free left R-modules by $\mathcal{F}_\tau$. The τ-torsion subfunctor of the identity functor on R-mod is denoted by $T_\tau(\_)$. A submodule N of a left R-module M is said to be *τ-pure* in M if and only if M/N is τ-torsion-free. The set R-tors is a complete lattice when we define $\tau \leq \tau'$ if and only if $\mathcal{T}_\tau \subseteq \mathcal{T}_{\tau'}$ (or equivalently if and only if $\mathcal{F}_\tau \supseteq \mathcal{F}_{\tau'}$) and when, for any subset U of R-tors, $\bigwedge U$ is characterized by $\mathcal{T}_{\wedge U} = \bigcap\{\mathcal{T}_\tau | \tau \in U\}$. For each $M \in R$-mod there is a unique largest element of R-tors relative to which M is torsion-free. We denote this theory by $\chi(M)$.

Let $\tau \in R$-tors. A nonzero left R-module M is said to be *τ-cocritical* if and only if M is τ-torsion-free and, for any nonzero submodule N of M, M/N is τ-torsion. Such modules are discussed, under various names, in [2] and [8]. Every nonzero submodule of a τ-cocritical left R-module is τ-cocritical and every τ-cocritical left R-module is uniform. Moreover, a left R-module M is τ-cocritical for some torsion theory τ if and only if it is $\chi(M)$-cocritical.

It is not true that every $\tau \in R$-tors is of the form $\chi(M)$ for some τ-cocritical left R-module M. If a torsion theory τ can be so represented then it is said to be *prime* [2]. Indeed, if τ is prime then $\tau = \chi(M)$ for every τ-cocritical left R-module M. We call the set of all prime torsion theories on R-mod the *left spectrum* of R and denote it by R-sp. If $\tau \in R$-tors we denote by pgen(τ) the set of all prime torsion theories $\pi \in R$-sp satisfying $\pi \geq \tau$. (These are the *prime generalizations* of τ.) It is not always true that, for a given torsion

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theory τ, we have $\tau = \bigwedge \mathrm{pgen}(\tau)$. When this happens τ is said to be *semiprime*. If $\tau = \bigwedge U$ for some $U \subseteq \{\chi(M) | M \in R\text{-mod is } \tau\text{-cocritical}\}$ then τ is said to be *strongly semiprime*. The rings over which every torsion theory is semiprime and those over which every torsion theory is strongly semiprime are characterized in [6]. The latter are precisely the left seminoetherian rings, i.e. those rings having left Gabriel dimension.

Let M be a left R-module. We define the *support*, $\mathrm{supp}(M)$, of M to be $\mathrm{supp}(M) = \{\pi \in R\text{-sp} | M \notin \mathcal{T}_\pi\}$. See [1] for details. We define the *assassin* $\mathrm{ass}(M)$ of M to be $\mathrm{ass}(M) = \{\pi \in R\text{-sp} | \text{there exists a } \pi\text{-cocritical submodule of } M\}$. See [1], [2], [8]. For any submodule N of M, $\mathrm{ass}(N) \subseteq \mathrm{ass}(M) \subseteq \mathrm{ass}(N) \cup \mathrm{ass}(M/N)$. Moreover, if M is π-cocritical for some $\pi \in R$-sp then $\mathrm{ass}(N) = \{\pi\}$ for any nonzero submodule N of M.

1. Modules which are both τ-artinian and τ-noetherian. Let $\tau \in R$-tors. Following Manocha [5], we say that a left R-module M is *τ-artinian* [resp. *τ-noetherian*] if and only if the set of all τ-pure submodules of M satisfies the descending chain condition [resp. the ascending chain condition]. The class of all τ-artinian [resp. τ-noetherian] left R-modules is a Serre class [5] and hence so is the class of all left R-modules which are both τ-artinian and τ-noetherian. We denote this class by $\mathfrak{A}_\tau$. Then $\mathcal{T}_\tau \subseteq \mathfrak{A}_\tau$ and for any left R-module M, $M \in \mathfrak{A}_\tau$ if and only if $M/T_\tau(M) \in \mathfrak{A}_\tau$.

(1.1) **Proposition.** *If* $\tau, \tau' \in R$-tors *then*:

(1) $\tau \leq \tau' \Rightarrow \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\tau'}$.

(2) $\mathfrak{A}_{\tau \wedge \tau'} = \mathfrak{A}_\tau \cap \mathfrak{A}_{\tau'}$.

Proof. (1) Let $M \in \mathfrak{A}_\tau$. If $M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of τ'-pure submodules of M then $M/M_i \in \mathcal{F}_{\tau'} \subseteq \mathcal{F}_\tau$ for each index i and so there exists an index k for which $M_k = M_{k+1} = \dots$. Thus M is τ'-noetherian. Similarly, M is τ'-artinian and so $M \in \mathfrak{A}_{\tau'}$.

(2) By (1), $\mathfrak{A}_{\tau \wedge \tau'} \subseteq \mathfrak{A}_\tau \cap \mathfrak{A}_{\tau'}$. Conversely, let $M \in \mathfrak{A}_\tau \cap \mathfrak{A}_{\tau'}$ and let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of $(\tau \wedge \tau')$-pure submodules of M. Then $M/M_j \notin \mathcal{T}_\tau \cap \mathcal{T}_{\tau'}$ for each index j and so we can assume without loss of generality that there are an infinite number of indices j for which M/M_j is not τ-torsion. By throwing away all of the other links in the chain we can in fact assume that M/M_j is not τ-torsion for all indices j.

For each index j, let N_j be the submodule of M defined by $N_j/M_j = T_\tau(M/M_j)$. Then we have an ascending chain $N_1 \subseteq N_2 \subseteq \dots$ of τ-pure submodules of M which must therefore terminate at some index k, i.e. $N_k = N_{k+1} = \dots$. For each $j > k$, we have $M_j/M_k \subseteq N_j/M_k = N_k/M_k \in \mathcal{T}_\tau$. But $M_j/M_k \subseteq M/M_k \in \mathcal{F}_{\tau \wedge \tau'}$ and so we must have $M_j/M_k \notin \mathcal{T}_{\tau'}$ for all $j > k$. For each $j > k$ let N'_j be the submodule of N_k defined by $M_j = T_{\tau'}(N_k/M_j)$. Then we have the ascending chain $N'_k \subseteq N'_{k+1} \subseteq \dots$ of τ'-pure submodules of N_k. Since $M \in \mathfrak{A}_{\tau'}$, we have $N_k \in \mathfrak{A}_{\tau'}$ and so there exists an index h for which $N'_h = N'_{h+1} = \dots$. If $j > h$ then $M_j/M_h \subseteq N'_j/M_h = N'_h/M_h \in \mathfrak{T}_{\tau'}$ and so $M_j/M_h \in \mathfrak{T}_\tau \cap \mathfrak{T}_{\tau'} = \mathfrak{T}_{\tau\wedge\tau'}$. But $M_j/M_h \subseteq M/M_h \in \mathfrak{F}_{\tau\wedge\tau'}$ and so we must have $M_j = M_h$. Therefore M is $(\tau \wedge \tau')$-noetherian. A similar proof shows that M is $(\tau \wedge \tau')$-artinian and so $M \in \mathfrak{A}_{\tau\wedge\tau'}$. □

(1.2) **Corollary.** *The following conditions are equivalent for* $M \in R$-mod:
(1) $M \in \mathfrak{A}_{\chi(M)}$.
(2) $M \in \mathfrak{A}_\tau \cap \mathfrak{F}_\tau$ *for some* $\tau \in R$-tors.

Proof. (1) $\Longrightarrow$ (2): Trivial. (2) $\Longrightarrow$ (1): If $M \in \mathfrak{A}_\tau \cap \mathfrak{F}_\tau$ then $\chi(M) \geq \tau$ and so $M \in \mathfrak{A}_{\chi(M)}$ by Proposition 1.1(1). □

Goldman [3] has shown that $M \in \mathfrak{A}_\tau$ if and only if there exists a chain $T_\tau(M) = M_0 \subset M_1 \subset \dots \subset M_n = M$ of submodules of M having the property that M_{i+1}/M_i is τ-cocritical for each $0 \leq i < n$. Such a chain is called a *τ-chain* in M. Moreover, if $T_\tau(M) = M'_0 \subset M'_1 \subset \dots \subset M'_k = M$ is another τ-chain in M then $k = n$ and

$$\{\chi(M_{i+1}/M_i) \mid 0 \leq i < n\} = \{\chi(M'_{j+1}/M'_j) \mid 0 \leq j < k\}.$$

The integer n is therefore unique and is called the *τ-length* of M; it is denoted by $\mathrm{len}_\tau(M)$. The set $\{\chi(M_{i+1}/M_i) | 0 \leq i < n\}$ of prime torsion theories is called the set of *τ-invariants* of M and is denoted by $\mathrm{inv}_\tau(M)$. For each $M \in \mathfrak{A}_\tau$ we have

$$\mathrm{len}_\tau(M) = \mathrm{len}_\tau(M/T_\tau(M)) \quad \text{and} \quad \mathrm{inv}_\tau(M) = \mathrm{inv}_\tau(M/T_\tau(M)).$$

Moreover, Goldman [3, Corollary 2.8] showed that if $M \in \mathfrak{A}_\tau$ then $\mathrm{inv}_\tau(M) = \mathrm{pgen}(\tau) \cap \mathrm{supp}(M)$. Thus, if $\tau \leq \tau'$ we have $\mathrm{inv}_\tau(M) \supseteq \mathrm{inv}_{\tau'}(M)$.

For each $M \in R$-mod and each $\tau \in R$-tors let $W(M, \tau)$ be the set of all prime torsion theories π satisfying both of the following conditions:

(1) π is a minimal element of $\mathrm{pgen}(\tau)$;
(2) π is a maximal element of $\mathrm{supp}(M)$.

(1.3) **Proposition.** *If* $\tau \in R$-tors *and* $M \in \mathfrak{A}_\tau$ *then* $\mathrm{inv}_\tau(M) = W(M, \tau)$.

Proof. We have already noted that $\mathrm{inv}_\tau(M) = \mathrm{pgen}(\tau) \cap \mathrm{supp}(M) \supseteq W(M, \tau)$. Conversely, if $\pi \in \mathrm{inv}_\tau(M)$ and if $\tau' \in R$-tors satisfies $\tau' > \pi$ we want to show that $\tau' \notin \mathrm{supp}(M)$. Assume the contrary. Then $0 \neq N = M/T_{\tau'}(M) \in \mathfrak{F}_{\tau'} \subseteq \mathfrak{F}_\pi \subseteq \mathfrak{F}_\tau$. Moreover, $M \in \mathfrak{A}_\tau$ implies that $N \in \mathfrak{A}_\tau$. If $0 = N_0 \subset \dots \subset N_k = N$ is a τ-chain in N then N_1 is a τ-cocritical submodule of N and so $\chi(N_1)$ is a minimal element of $\mathrm{pgen}(\tau)$ for if $\pi'' \in \mathrm{pgen}(\tau)$ and $\pi'' \leq \chi(N_1)$, then $N_1 \in \mathfrak{F}_{\chi(N_1)} \subseteq \mathfrak{F}_{\pi''}$. On the other hand, for any nonzero submodule N' of N_1, $N_1/N' \in \mathfrak{T}_\tau \subseteq \mathfrak{T}_{\pi''}$. Therefore N_1 is also π''-cocritical and so $\pi'' = \chi(N_1)$. But $\chi(N_1) \geq \chi(N) \geq \tau' > \pi \in \mathrm{pgen}(\tau)$, a contradiction. Therefore we must have $\tau' \notin \mathrm{supp}(M)$ and so π is a maximal element of $\mathrm{supp}(M)$.

We now want to show that π is a minimal element of pgen(τ). If $T_\tau(M) = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a τ-chain in M then there exists an index k, $0 \leq k < n$, for which $\pi = \chi(M_{k+1}/M_k)$, where M_{k+1}/M_k is a τ-cocritical left R-module. But we have already seen that this suffices to prove that π is minimal in pgen(τ). Therefore $\pi \in W(M, \tau)$ and so we have that $\mathrm{inv}_\tau(M) = W(M, \tau)$. □

(1.4) **Corollary.** *If $M \in \mathfrak{A}_\tau$ for some torsion theory τ then $M \in \mathfrak{T}_\tau$ if and only if $W(M, \tau) = \varnothing$.*

(1.5) **Proposition.** *Let $M \in R$-mod and let $\tau \in R$-tors be semiprime. Then the following conditions are equivalent:*

(1) $M \in \mathfrak{A}_\tau$.

(2) *$W(M, \tau)$ is a finite set and $M \in \mathfrak{A}_\pi$ for each $\pi \in W(M, \tau)$.*

Proof. (1) $\Longrightarrow$ (2): By Proposition 1.3, (1) implies that $W(M, \tau) = \mathrm{inv}_\tau(M)$ and the latter is a finite set. By Proposition 1.1(1), $M \in \mathfrak{A}_\pi$ for each $\pi \geq \tau$ and so in particular for each $\pi \in W(M, \tau)$.

(2) $\Longrightarrow$ (1): Let $U = \mathrm{pgen}(\tau)\backslash W(M, \tau)$. Then $M \in \mathfrak{T}_{\wedge U} \subseteq \mathfrak{A}_{\wedge U}$. Moreover, since τ is semiprime we have $\tau = \bigwedge \mathrm{pgen}(\tau) = [\bigwedge U] \wedge [\bigwedge W(M, \tau)]$. Since $W(M, \tau)$ is a finite set and since $M \in \mathfrak{A}_\pi$ for each $\pi \in W(M, \tau)$, we have $M \in \mathfrak{A}_\tau$ by Proposition 1.1(2). □

2. **Modules having finite intrinsic length.** In Corollary 1.2 we saw the importance of asking when a left R-module M belongs to $\mathfrak{A}_{\chi(M)}$. Goldman [3] calls modules having this property modules with *finite intrinsic length*.

(2.1) **Proposition.** *If $0 \neq M \in \mathfrak{A}_{\chi(M)}$ then:*

(1) $\varnothing \neq \mathrm{ass}(M) = \mathrm{inv}_{\chi(M)}(M)$;

(2) $\mathrm{ass}(M)$ *is a finite set*;

(3) *there exists a large submodule of M of the form $\bigoplus_{i=1}^n N_i$ where the N_i are cyclic cocritical submodules of M*;

(4) *M has finite uniform dimension*;

(5) *to each $\pi \in \mathrm{ass}(M)$ we can associate a submodule N_π of M such that the following conditions are satisfied*:

(i) $\mathrm{ass}(M/N_\pi) = \{\pi\}$,

(ii) $\mathrm{ass}(N_\pi) = \mathrm{ass}(M)\backslash\{\pi\}$;

(iii) $0 = \bigcap\{N_\pi \mid \pi \in \mathrm{ass}(M)\}$ *and this intersection is reduced*;

(6) $\chi(M) = \bigwedge \mathrm{ass}(M)$;

(7) *$\chi(M)$ is strongly semiprime.*

Proof. We will first prove that for any left R-module $M' \in \mathfrak{F}_{\chi(M)} \cap \mathfrak{A}_{\chi(M)}$ we have $\mathrm{ass}(M') \subseteq \mathrm{inv}_{\chi(M)}(M')$. This will prove one inclusion of (1). The proof of the reverse inclusion will be delayed until later. We will proceed by induction on $\mathrm{len}_{\chi(M)}(M')$. If $\mathrm{len}_{\chi(M)}(M') = 1$ then M' is $\chi(M)$-cocritical and so $\mathrm{ass}(M') = \{\chi(M')\} = \mathrm{inv}_{\chi(M)}(M')$. Now assume that $\mathrm{len}_{\chi(M)}(M') = n$

and that the result has been established for all $\chi(M)$-torsion-free modules the $\chi(M)$-length of which is less than n. Let $0 = M_0' \subset M_1' \subset \ldots \subset M_n' = M'$ be a $\chi(M)$-chain in M'. Then M'/M_1' is $\chi(M)$-torsion-free [3] and $\operatorname{len}_{\chi(M)}(M'/M_1') = n - 1$. Moreover, $\operatorname{inv}_{\chi(M)}(M') = \{\chi(M_1)\} \cup \operatorname{inv}_{\chi(M)}(M'/M_1')$. Let $\pi \in \operatorname{ass}(M')$. There then exists a submodule N of M' for which $\pi = \chi(N)$. If $N \cap M_1' \neq 0$ then $N \cap M_1'$ is π-cocritical and so $\pi = \chi(N \cap M_1') = \chi(M_1') \in \operatorname{inv}_{\chi(M)}(M')$. If $N \cap M_1' = 0$ then $N \cong [N + M_1']/M_1' \subseteq M'/M_1'$ and so $\pi \in \operatorname{ass}(M'/M_1')$. By induction we then have $\pi \in \operatorname{inv}_{\chi(M)}(M'/M_1') \subseteq \operatorname{inv}_{\chi(M)}(M')$.

In particular, we have $\operatorname{ass}(M) \subseteq \operatorname{inv}_{\chi(M)}(M)$ which is one direction of (1). Since $\operatorname{inv}_{\chi(M)}(M)$ is a finite set, we immediately have (2).

The module M certainly has at least one cocritical submodule N (i.e. N is τ-cocritical for some $\tau \in R$-tors), namely the first nonzero link in a $\chi(M)$-chain in M. Therefore $\varnothing \neq \operatorname{ass}(M)$. If $0 \neq x \in N$ then Rx is also cocritical and so without loss of generality we can assume that N is a cyclic left R-module. Consider the set of all submodules of M of the form $\bigoplus_{i\in\Lambda} N_i$ where the N_i are cyclic cocritical submodules of M. This set is clearly inductive and so by Zorn's lemma it contains a maximal element $M' = \bigoplus_{i\in\Omega} N_i$. We claim that M' is a large submodule of M. Indeed, if not then there exists a nonzero submodule M'' of M for which $M'' \cap M' = 0$. By Proposition 1.1, $M'' \in \mathfrak{A}_{\chi(M)}$ and so M'' has a nonzero cocritical submodule N'', which we can again assume to be cyclic. Therefore $M' \oplus N''$ is strictly larger than M', contradicting the maximality of M'. Thus M' is large in M. We now want to show that the set Ω is finite. Assume not. Then without loss of generality we can assume that Ω contains the set of natural numbers. For each natural number i, let $Y_i = \bigoplus_{j=1}^{i} N_j$. Then for each index i, M'/Y_i is isomorphic to a submodule of M' and so is $\chi(M)$-torsion-free. Hence we have an infinite ascending chain $Y_1 \subset Y_2 \subset \ldots$ of $\chi(M)$-pure submodules of M'. But if $M \in \mathfrak{A}_{\chi(M)}$ then $M' \in \mathfrak{A}_{\chi(M)}$, a contradiction. Therefore Ω is a finite set and so we have proven (3). As an immediate consequence of this we also have (4).

Now let $\pi \in \operatorname{ass}(M)$ and consider the class of all submodules N' of M for which $\pi \notin \operatorname{ass}(N')$. This set is clearly nonempty and by [8, Proposition 3.1] it is inductive. Therefore by Zorn's lemma there is a maximal such module N_π which clearly satisfies (5)(ii).

Let $\pi' \in \operatorname{ass}(M/N_\pi)$. Then there exists a π'-cocritical submodule M'/N_π of M/N_π. Thus

$$\operatorname{ass}(M') \subseteq \operatorname{ass}(N_\pi) \cup \operatorname{ass}(M'/N_\pi) = \operatorname{ass}(N_\pi) \cup \{\pi'\}$$

and so, by the maximality of N_π, we have $\pi' = \pi$. This establishes (5)(i).

Now let $N' = \bigcap\{N_\pi \mid \pi \in \operatorname{ass}(M)\}$. Then $\operatorname{ass}(N') \subseteq \operatorname{ass}(N_\pi)$ for all $\pi \in \operatorname{ass}(M)$ and so $\operatorname{ass}(N') = \varnothing$. But $M \in \mathfrak{A}_{\chi(M)}$ implies that $N' \in \mathfrak{A}_{\chi(M)}$ and so by what we proved above this implies that $N' = 0$. If U is a proper subset of ass(M) and if $N'' = \bigcap\{N_\pi \mid \pi \in U\}$, then ass$(N'') = $ ass$(M)\setminus U \neq \varnothing$ and so $N'' \neq 0$. This proves (5)(iii) and so completes the proof of (5).

If $\pi \in$ ass(M) then $\pi \in \mathrm{inv}_{\chi(M)}(M) \subseteq \mathrm{pgen}(\chi(M))$. Therefore $\bigwedge \mathrm{ass}(M) \geq \chi(M)$. On the other hand, by (5) there exists a canonical monomorphism $M \longrightarrow M' = \bigoplus\{M/N_\pi \mid \pi \in \mathrm{ass}(M)\}$. Then $\chi(M) \geq \chi(M') = \bigwedge \mathrm{ass}(M)$ by [2, Proposition 5.4] and so $\chi(M) = \bigwedge \mathrm{ass}(M)$, proving (6).

In particular, (6) implies that $\chi(M)$ is semiprime. Since ass$(M) \subseteq \mathrm{inv}_{\chi(M)}(M)$, each $\pi \in$ ass(M) is of the form $\chi(N)$ where N is a $\chi(M)$-cocritical left R-module. This implies that $\chi(M)$ is strongly semiprime, proving (7).

Finally, we return to prove the other direction of (1). Let $\pi \in \mathrm{inv}_{\chi(M)}(M)$. Then $\pi \geq \chi(M) = \bigwedge \mathrm{ass}(M)$. By (2), ass$(M)$ is a finite set of prime torsion theories and so by [3, Lemma 3.5] there exists a $\pi' \in$ ass(M) for which $\pi \geq \pi'$. But by Proposition 1.3, π and π' are both maximal elements of supp(M) and so we must have $\pi = \pi'$. Therefore $\pi \in$ ass(M) and so $\mathrm{inv}_{\chi(M)} \subseteq$ ass(M), proving equality. □

(2.2) **Corollary.** *The following conditions are equivalent for* $0 \neq M \in R$-mod:

(1) $M \in \mathfrak{A}_{\chi(M)}$.

(2) (i) $\chi(M)$ *is strongly semiprime*;

(ii) $W(M, \chi(M))$ *is a finite set*;

(iii) $M \in \mathfrak{A}_\pi$ *for each* $\pi \in W(M, \chi(M))$.

Proof. By Propositions 2.1 and 1.5. □

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