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BOUNDS FOR NEARLY BEST APPROXIMATIONS

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ABSTRACT. Let X be a uniformly convex space and ψ be the inverse function of the modulus of convexity $\delta(\cdot)$. Assume here that ψ is a concave function. Let V be a linear subspace of X and let f in X be such that $\|f\| = 1 = \min\{\|f - v\| : v \in V\}$. Then for $0 < \delta \leq 1$ and for v in V with $\|f - v\| \leq 1 + \delta$, it follows that $\|v\| \leq K \cdot \psi(\delta)$.

Let T be a compact Hausdorff-space and V a finite-dimensional subspace of $C(T, X)$. When V has the interpolation property (P_m) with $\dim V = m \cdot \dim X$, then the same type of estimate as above holds.

Let X be a uniformly convex normed linear space [1], i.e., for each ϵ with $0 < \epsilon \leq 2$ there exists a $\delta(\epsilon) > 0$ such that $x, y \in X$, $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| > \epsilon$ imply $\|(x + y)/2\| \leq 1 - \delta(\epsilon)$. The function $\delta(\cdot)$ is called the modulus of convexity of X. Without loss of generality we shall always assume that $\delta(\cdot)$ is monotone nondecreasing. Then an inverse function ψ can be defined by

$$(1) \qquad \psi(\delta_0) := \sup\{\epsilon : 0 < \epsilon \leq 2, \ \delta(\epsilon) < \delta_0\}$$

for $\delta_0 > 0$. Obviously, ψ is monotone nondecreasing. From $\delta(\epsilon) \leq \epsilon/2$ it follows that $\psi(\delta_0) > 0$ for $\delta_0 > 0$.

One can replace $\delta(\cdot)$ by a monotone increasing convex function $\delta_1(\cdot)$, such that $0 < \delta_1(\epsilon) < \delta(\epsilon)$ for $0 < \epsilon \leq 2$ and $1 \leq \liminf_{\epsilon \to 0} \delta(\epsilon)/\delta_1(\epsilon) < \infty$. Then ψ is concave and continuous.

Let V be a subspace of X, and let f be in X such that $\|f\| = 1$ and 0 is the best approximation for f by elements of V. A question of some practical interest is that of how fast the "nearly best approximations" v in V, with $\|f - v\| \leq 1 + \delta$, approach 0 when $\delta \to 0$.

This note considers also the analogous question for subspaces V of $C(T, X)$, T compact, and gives estimates for $\|v\|$ in terms of the function ψ.

Theorem 1. *The diameter $D(C)$ of every convex subset C of the spherical shell $R(\delta) := \{x \in X : 1 - \delta \leq \|x\| \leq 1\}$ is $\leq \psi(\delta)$.*

A result of this type was given by Fan and Glicksberg [2], but they did not relate the bound on $D(C)$ to the modulus of convexity.

Proof. From (1) it follows that $\delta(\epsilon) \geq \delta_0$ for $\epsilon > \psi(\delta_0)$. So, $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|(x + y)/2\| > 1 - \delta$ imply $\|x - y\| \leq \psi(\delta)$.

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Let C be a convex subset of $R(\delta)$. Then for $x, y \in C$, $x \neq y$, the segment $[x, y]$ is in $R(\delta)$. Define $y_\theta := x + \theta(y - x)$. Then $\|x\| \leq 1$, $\|y_\theta\| \leq 1$, and $\|(x + y_\theta)/2\| \geq 1 - \delta$ for $0 \leq \theta \leq 1$. Since X is uniformly convex this last inequality is strict for all θ with at most one exception θ_0. Thus we obtain $\|x - y_\theta\| \leq \psi(\delta)$ for all $\theta \neq \theta_0$ and by continuity also for θ_0. Since x, y in C are arbitrarily chosen, $D(C) \leq \psi(\delta)$ is proved.

Theorem 2. *Let V be a linear subspace of X, let f be in X such that $\|f\| = 1 = \min\{\|f - v\| : v \in V\}$. Then for $0 < \delta \leq 1$ and all $v \in V$ with $\|f - v\| \leq 1 + \delta$ it follows that $\|v\| \leq 2\psi(\delta)$.*

Proof. The set $C := \{v \in V : \|f - v\| \leq 1 + \delta\}$ is a convex subset of the shell $\{x \in X : 1 \leq \|x - f\| \leq 1 + \delta\}$. Using Theorem 1 to estimate the diameter of C, we obtain

$$\|v\| = \|v - 0\| \leq D(C) \leq (1 + \delta)\psi(1 - 1/(1 + \delta))$$
$$= (1 + \delta)\psi(\delta/(1 + \delta)) \leq 2\psi(\delta).$$

Let P_V be the metric projection on V, i.e. the mapping which assigns to each f in X its best approximation $P_V(f)$ by elements of V. It is well known that P_V is uniformly continuous on bounded sets [5, p. 17]. From Theorem 2 we can obtain bounds for the modulus of continuity of P_V.

Corollary 1. *Let V be a linear subspace of X. Let f, g in X be such that $2\|f - g\| \leq E(f) := \min\{\|f - v\| : v \in V\}$. Then*

$$\|P_V(f) - P_V(g)\| \leq 2E(f)\psi(2\|f - g\|/E(f)).$$

Proof. Without loss of generality we assume $P_V(f) = 0$, so that $E(f) = \|f\|$. Using $\|P_V(g) - g\| \leq \|f - g\| + E(f)$ we can estimate

$$E(f) \leq \|P_V(g) - f\| \leq \|P_V(g) - g\| + \|f - g\| \leq E(f) + 2\|f - g\|.$$

It follows that

$$1 \leq \|P_V(g) - f\|/E(f) \leq 1 + 2\|f - g\|/E(f),$$

and by Theorem 2,

$$\|P_V(g) - P_V(f)\| \leq 2E(f) \cdot \psi(2\|f - g\|/E(f)).$$

Let T be a compact Hausdorff space and $C(T, X)$ be the space of continuous functions $f : T \to X$ provided with the maximum norm $\|f\| := \max\{\|f(t)\|_X : t \in T\}$. A subspace V of $C(T, X)$ is said to have the interpolation property (P_m) if for every m distinct points $t_1, \ldots, t_m$ in T and elements $y_1, \ldots, y_m$ in X there exists v in V such that $v(t_i) = y_i$ for $i = 1, \ldots, m$ [6, p. 201]. When the real dimensions are in the relation $\dim V = m \cdot \dim X$, then there exists exactly one such v, and each function v in V which vanishes at m distinct points on T vanishes identically.

The following theorem is analogous to Theorem 2.

Theorem 3. *Let V be a linear subspace of $C(T, X)$ which has property*

(P_m) *with* $\dim V = m \cdot \dim X$. *Let* f *in* $C(T, X)$ *be such that* $\|f\| = 1 = \min\{\|f - v\| : v \in V\}$. *Then there exist numbers* $K_1 > 0$, $K_2 \geq 1$ *depending on* f *and* V *such that for all* v *in* V *with* $\|f - v\| \leq 1 + \delta$ *it follows that*

$$\|v\| \leq K_1 \psi(K_2 \delta). \tag{2}$$

If ψ *is a concave function, then*

$$\|v\| \leq K_3 \psi(\delta). \tag{3}$$

Proof. Let n be the dimension of V over the real field. According to [6, p. 202] the element 0 is a best approximation for f by elements of V if and only if there exist extremal points $x_1^*, \ldots, x_h^*$ of the unit ball $\{x^* \in X^*: \|x^*\| \leq 1\}$ of the dual space X^*, points $t_1, \ldots, t_h$ in T and positive numbers λ_j with $\Sigma_{j=1}^h \lambda_j = 1$ such that

$$\sum_{j=1}^{h} \lambda_j x_j^*(v(t_j)) = 0 \quad \text{for each } v \text{ in } V, \tag{4}$$

$$x_j^*(f(t_j)) = \|f\| = 1 \quad \text{for } j = 1, \ldots, h. \tag{5}$$

The number h is in the range $m + 1 \leq h \leq n + 1$.

Since V has property (P_m) with $\dim V = m \cdot \dim X$ from $v \in V$ and $v(t_j) = 0$ for $j = 1, \ldots, h$ it follows that $v \equiv 0$. Hence $\max\{\|v(t_j)\| : j = 1, \ldots, h\}$ is a norm on V. Since V has finite dimension this norm is equivalent to the original one, i.e., there is a constant K_4 so that

$$\|v\| \leq K_4 \max\{\|v(t_j)\| : j = 1, \ldots, h\} \quad \text{for } v \text{ in } V. \tag{6}$$

From $\|f(t_j) - v(t_j)\| \leq 1 + \delta$ it follows that $|x_j^*(f(t_j) - v(t_j))| \leq 1 + \delta$. For each fixed index k in $1 \leq k \leq h$ we obtain from (4) and (5)

$$\sum_{j \neq k} \lambda_j + \lambda_k x_k^*(v(t_k)) = \sum_{j \neq k} \lambda_j x_j^*(f(t_j) - v(t_j))$$

$$\leq \sum_{j \neq k} \lambda_j |x_j^*(f(t_j) - v(t_j))| \leq \sum_{j \neq k} \lambda_j (1 + \delta),$$

and consequently

$$\lambda_k x_k^*(v(t_k)) \leq \left(\sum_{j \neq k} \lambda_j \right) \delta. \tag{7}$$

The number $K_5 := \max\{\Sigma_{j \neq k}(\lambda_j / \lambda_k) : k = 1, \ldots, h\}$ depends on f and V, but not on v. So we obtain from (7)

$$x_k^*(v(t_k)) \leq K_5 \cdot \delta \quad \text{for } k = 1, \ldots, h.$$

For both points, $x_k = 0$ and $x_k = v(t_k)$, we have $\|f(t_k) - x_k\| \leq 1 + \delta$ and $x_k^*(x_k) \leq K_5 \cdot \delta$, hence by (5) $x_k^*(f(t_k) - x_k) \geq 1 - K_5 \delta$. Consequently $(f(t_k) - x_k)/(1 + \delta)$ is in the convex subset $C := \{x \in X : \|x\| \leq 1, x_k^*(x) \geq$

$(1 - K_5\delta)/(1+\delta)\}$ of the spherical shell $\{x \in X : 1 - (K_5 + 1) \cdot \delta/(1+\delta) \leq \|x\| \leq 1\}$. Using the estimate of Theorem 1 for the diameter $D(C)$ we obtain

$$\|v(t_k)\| \leq (1+\delta)D(C) \leq (1+\delta)\psi\left(\frac{(K_5+1)\delta}{1+\delta}\right) \leq 2\psi((K_5+1)\delta)$$

for $k = 1, \ldots, h$. Together with (6) this yields the estimate (2). If ψ is a concave function then we can use $\psi(\lambda\delta) \leq \lambda\psi(\delta)$ for $\lambda \geq 1$ to obtain (3).

For X a real Hilbert space one can choose $\psi(\delta) = \delta$ if $\dim X = 1$ and $\psi(\delta) = 2\delta^{1/2}$ if $\dim X \geq 2$. We note that $\mathbf{C}$ is norm-isomorphic to the Euclidean $\mathbf{R}^2$. The space V of the polynomials of degree $\leq n$ has the interpolation property (P_{n+1}) in the real as well as in the complex case. So we obtain from Theorem 3 the following

Corollary 2. *Let T be a compact subset of $\mathbf{R}$ (or $\mathbf{C}$) with at least $n+2$ points and let V be the space of polynomials of degree $\leq n$ restricted to T. Let f be in $C(T, \mathbf{R})$ (or $C(T, \mathbf{C})$) such that $\|f\| = 1 = \min\{\|f - v\|, v \in V\}$. Then there exists a number K dependent on T, n and f, such that for all v in V with $\|f - v\| \leq 1 + \delta$ it follows that $\|v\| \leq K \cdot \delta$ (or $\|v\| \leq K\delta^{1/2}$).*

In the real case this is a result of Freud [3]. The complex case improves a result of Poreda, who proved in [4] only $\|v\| = O(\delta^{\beta})$ for $0 < \beta < 1/2$.

Now we show that the estimate (3) is sharp in the sense that the function ψ may not be replaced by another one ψ_1 such that $\psi_1(\delta)/\psi(\delta) \to 0$ as $\delta \to 0$. We make the hypothesis that ψ is concave and sharp in the following sense: There exists a constant $K > 0$ such that for all x in X and $x^* \in X^*$ with $\|x^*\| = 1 = \|x\| = x^*x$, from $\|y\| = 1 + \delta$ and $x^*(x - y) = 0$ it follows that $\|y - x\| \geq K\psi(\delta)$.

We note that Hilbert-spaces have this property, when ψ is specified as before Corollary 2. So the estimates of the corollary are sharp.

To prove the sharpness of (3) we proceed in the following way. Let V be a subspace of $C(T, X)$ as in Theorem 3. We construct a suitable f in $C(T, X)$ which fulfills the hypotheses of Theorem 3 such that for all $\delta > 0$ there exists v in V such that $\|f - v\| \leq 1 + \delta$ and $\|v\| \geq K\psi(\delta)$.

Let $t_1, \ldots, t_{m+1}$ be different points of T. The mapping $v \to (v(t_1), \ldots, v(t_{m+1}))$ carries V onto an n-dimensional subspace of the $(m+1)$-fold product $W := X \times \cdots \times X$, which has dimension $(m+1) \cdot \dim X > n$. So there exists a nontrivial linear functional w^* on W which vanishes on the image of V. Hence there exist x_j^* in X^* and real λ_j such that

$$\sum_j \lambda_j x_j^*(v(t_j)) = 0 \tag{8}$$

for all v in V. By suitable normalization we can reach $\|x_j^*\| = 1$, $\lambda_j \geq 0$, $\sum_j \lambda_j = 1$.

Since X has finite dimension there exist x_j in X so that $\|x_j\| = 1 =$

$x_j^* x_j$. We put $f(t_j) = x_j$ and extend f to an element of $C(T, X)$ such that for an $\eta > 0$

$$\|f(t) - v(t)\| \le \max\{\|f(t_j) - v(t_j)\| : j = 1, \ldots, m+1\}$$

holds for all t in T and v in V with $\|v\| \le \eta$. We omit the lengthy but elementary details of this construction. For this f we have $\|f\| = 1 = \min\{\|f - v\|: v \in V\}$.

If $X = \mathbf{R}$, it follows from $\|f - v\| = 1 + \delta$ that $|f(t_j) - v(t_j)| = 1 + \delta$ for at least one j, and so $\|v\| \ge |v(t_j)| = \delta$.

In case $\dim X \ge 2$ we construct a $v \neq 0$ in V with

$$x_j^* v(t_j) = 0 \quad \text{for all } j. \tag{9}$$

Let $v_1, \ldots, v_n$ be a basis of V, then (9) leads with $v = \sum \alpha_\nu v_\nu$ to the system of equations

$$\sum_\nu \alpha_\nu x_j^* v_\nu(t_j) = 0, \qquad j = 1, \ldots, m+1, \tag{10}$$

which has a nontrivial solution $\alpha_1, \ldots, \alpha_n$, since the rank of the matrix of (10) is at most $n - 1$ because of $n \ge m + 1$ and (8). Therefore a $v \neq 0$ in V with (9) exists.

If $\|f - \lambda v\| = 1 + \delta$ then $\|f(t_j) - \lambda v(t_j)\| = 1 + \delta$ for at least one j. From this it follows $\|\lambda v(t_j)\| \ge K\psi(\delta)$ by hypothesis and so $\|\lambda v\| \ge K\psi(\delta)$.

From Theorem 3 one can obtain a bound for the modulus of continuity of the metric projection similar to that of Corollary 1. It may be noted that the bound of Corollary 1 is not sharp in general. For a Hilbert-space of dimension ≥ 2 it yields $\|P_V(f) - P_V(g)\| = O(\|f - g\|^{1/2})$ which is less sharp than the well-known estimate $\|P_V(f) - P_V(g)\| \le \|f - g\|$.

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