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COMPACTNESS OF SETS OF INTEGRABLE FUNCTIONS WITH VALUES IN A BANACH SPACE

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ABSTRACT. The present paper gives direct proofs of compactness (in a pairing topology) of bounded sets of strongly integrable functions with values in weakly compact subsets of a Banach space. The pairing in question is that one defined by the integral of the scalar product of two strongly integrable functions with values in the Banach space and its dual, respectively.

Notation. The concept of measurability of a function from an abstract measure space Ω, with measure μ, into a Banach space X, is used in the sense of Dunford and Schwartz [1, Chapter III]. This pertains also to the concept of integrability.

Let $q \in [1, \infty]$, $p \in [1, \infty]$, $1/p + 1/q = 1$. The symbols $L_p(\Omega, X)$ and $L_q(\Omega, X^*)$ denote the spaces of p- and q-integrable functions with values in X, and in X^* (the dual of X), respectively (if $p = \infty$ or $q = \infty$, the corresponding space consists of (strongly) measurable essentially bounded functions). The pairing we shall apply throughout this paper is $\int\langle z, w\rangle\, d\mu$, $z \in L_p(\Omega, X)$, $w \in L_q(\Omega, X^*)$. This pairing induces a pairing topology in each of these spaces, for which we reserve the name *the pairing topology*. (See [2].)

At last, note that a set is compact if each net contains a convergent subnet, a set is countably compact if each sequence contains a convergent subnet, and a set is sequentially compact if each sequence contains a convergent subsequence. This may be taken as definitions of these three concepts.

Theorem 1. *Let X be a separable Banach space, X^* its topological dual. Let Ω be a finite measure space. Let A be a closed convex set in X with the property that bounded subsets of A are relatively weakly compact. Let H be a norm-bounded subset of the space $L_p(\Omega, X)$ such that each $f \in H$ has the property that $f(t) \in A$ for all $t \in \Omega$. If $p \in \langle 1, \infty]$, then H is a relatively compact subset in $L_p(\Omega, X)$ in the pairing topology. If $p = 1$, then H is relatively compact in the pairing topology if H is also equi-integrable (i.e. for all $\epsilon > 0$, there exists a $\delta > 0$, such that if $\mu(M) < \delta$, then $\int_M \|f\|\, d\mu < \epsilon$, for all $f \in H$, μ being the measure on Ω).*

Proof. The theorem holds for $X = A = R$ (where R is the set of scalars, real or complex); see for example Dunford and Schwartz [1, V.4.2, IV.8.1,

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IV.8.9, IV.8.5, V.6.1]. Let the set $N = \{0, 1, 2, \cdots\}$ have the measure ν defined by $\nu(n) = 1/2^n$. Then the theorem holds for the space $L_p(\Omega \times N, R)$, where the set $\Omega \times N$ is given the (Fubini-) product measure.

Given a net $\{z_\lambda\}_{\lambda \in \Gamma}$ in $H \subset L_p(\Omega, X)$, $p \in [1, \infty)$. We shall first prove the existence of a measurable function $z(t)$ with the property that there exists a subnet $\{z_{\lambda'}\}_{\lambda' \in \Gamma'}$ of $\{z_\lambda\}_{\lambda \in \Gamma}$ such that $\int \langle z_{\lambda'}(\cdot), gz^*\rangle\, d\mu \to \int \langle z(\cdot), gz^*\rangle\, d\mu$, for all $z^* \in X^*$, and all realvalued nonnegative measurable step functions g.

Let b_i, $i = 1, 2, 3, \ldots$, be a sequence of linear functionals from X^* forming a dense set in the unit ball of X^* in the weak* topology (the pairing topology induced by the pairing of X, X^*). Let $\{x_\lambda(\cdot, \cdot)\}$ be the net of functions from $L_p(\Omega \times N, R)$ defined by $x_\lambda(t, 0) = \|z_\lambda(t)\|$, $x_\lambda(t, i) = \langle z_\lambda(t), b_i\rangle$ for $i > 0$. The net $\{x_\lambda\}$ is obviously norm-bounded for $p \geq 1$. The net $\{x_\lambda\}$ is also equi-integrable when $p = 1$: For any given $\epsilon > 0$, there is an integer m such that $\int_{i>m} \int_\Omega |x_\lambda|\, d\mu\, d\nu < \epsilon/2$, uniformly in λ, since $\{z_\lambda\}$ is bounded, and $\int_{i \leq m} \int_\Omega |x_\lambda|\, \chi_E\, d\mu\, d\nu < \epsilon/2$, uniformly in λ, if E has a product measure which is small enough. For such E, $\int_N \int_\Omega |x_\lambda|\, \chi_E\, d\mu\, d\nu < \epsilon$ for all λ.

Now, there exists a subnet of $\{x_\lambda\}$ which is convergent in the pairing topology—since set $\{x_\lambda : \lambda \in \Gamma\}$ is relatively compact (see the remarks in the beginning of the proof). We assume that $\{x_\lambda\}$ itself converges in $L_p(\Omega \times N, R)$ in the pairing topology, to some limit $x \in L_p(\Omega \times N, R)$. Since x lies in the norm-closed convex hull of the set $\{x_\lambda : \lambda \in \Gamma\}$, there exists a sequence y_n converging to x in the L_p-norm, y_n being convex combinations $\sum_{i=1}^{M_n} a_i^n x_{\lambda_{n,i}}$ of elements $x_{\lambda_{n,i}}$ in the net. If we let $z_n = \sum_{i=1}^{M_n} a_i^n z_{\lambda_{n,i}}$, we see that $\langle z_n(t), b_i\rangle = y_n(t, i)$, for $i > 0$. Note we have $y_n(t, 0) \geq \|z_n(t)\|$, for all t.

By considering subsequences if necessary, we may assume that $y_n(\cdot, \cdot) \to x(\cdot, \cdot)$ a.e., that is, there is a null set $\check{N}$ in Ω, such that $y_n(t, i) \to x(t, i)$ for all $t \in \Omega' = \Omega \setminus \check{N}$, $i \in N$. For $t \in \Omega'$, $y_n(t, 0)$ is convergent, hence $\{z_n(t)\}$ is bounded for such t.

By weak* density of the b_i's, and relative weak compactness of $\{z_n(t)\}$, $\{z_n(t)\}$ is a weak Cauchy sequence. [To see this, let H_t be the weak closure of $\{z_n(t)\}$. $\{z_n(t)\}$ is a Cauchy sequence in the (relative) topology induced by the pairing X, linspan$\{b_i\}$. This topology is equivalent to the weak topology on H_t, [2, 16.7].] Hence $\{z_n(t)\}$ converges, by weak compactness, to an element $z(t)$ in X, this for all t in Ω'. We may assume that $z(t)$ is defined for all t in Ω. Since $y_n(t, 0) \geq \|z_n(t)\| \geq |\langle z_n(t), v\rangle|$, for all v in the unit ball in X^*, $\|z(t)\| \leq \lim y_n(t, 0) = x(t, 0)$ a.e. Now, $x(t, 0) \in L_p(\Omega, R)$, hence $z(\cdot)$ belongs to $L_p(\Omega, X)$, if we prove it to be measurable.

It suffices for the proof of measurability of $z(\cdot)$ to show that $\langle z(\cdot), z^*\rangle$ is measurable for all $z^* \in X^*$ (since X is separable, cf. Dunford and Schwartz [1, III.6.11]). But this follows from the fact that $\langle z_n(t), z^*\rangle \to \langle z(t), z^*\rangle$ a.e. for $z^* \in X^*$.

Now, if g is a realvalued nonnegative measurable step function, then $\{\int g z_\lambda(\cdot)\, d\mu\}$ is relatively weakly compact, as a consequence of the properties of A. Define $f \in L_q(\Omega \times N, R)$ as follows: $f(t, i) = g(t)$, $f(t, i') = 0$, $i' \neq i$. We know that

$$\int \langle z_\lambda(\cdot), g b_i\rangle d\mu = \int_N \int_\Omega f(\cdot\,, \cdot) x_\lambda(\cdot\,, \cdot)\, d\mu\, d\nu$$

$$\to \int_N \int_\Omega f(\cdot\,, \cdot) x(\cdot\,, \cdot)\, d\mu\, d\nu = \int \langle z(\cdot), g b_i\rangle\, d\mu.$$

Now $\int \langle z_\lambda(\cdot), g b_i\rangle\, d\mu = \langle \int g z_\lambda(\cdot)\, d\mu, b_i\rangle$. By weak* density of the b_i's and relative weak compactness of $\{\int g z_\lambda(\cdot)\, d\mu\}$ (and [2, 16.7]),

$$\int \langle z_\lambda(\cdot), g z^*\rangle\, d\mu = \left\langle \int g z_\lambda(\cdot)\, d\mu, z^*\right\rangle \to \left\langle \int g z(\cdot)\, d\mu, z^*\right\rangle$$

$$= \int \langle z(\cdot), g z^*\rangle\, d\mu \quad \text{for } all\ z^* \in X^*.$$

Let Z be the linear span of the set of functions gz^*, $z^* \in X^*$, g a nonnegative measurable step function. Then $\int \langle z_\lambda(\cdot), w\rangle\, d\mu \to \int \langle z(\cdot), w\rangle\, d\mu$ for all $w \in Z$.

Furthermore, if $p > 1$, this holds even for $w \in L_q(\Omega, X^*)$, since the net $\{z_\lambda\}$ is bounded and Z is norm-dense in $L_q(\Omega, X^*)$. For $p = 1$, we first observe that functions of the type $\sum_{i=1}^\infty z_i^* \chi_{A_i}$, ($z_i^* \in X^*$, $\{A_i\}$ a partition of Ω into measurable sets), make up a set Z^* which is dense in $L_\infty(\Omega, X^*)$ (in the norm $\|\ \|_\infty$). We know that $\int \langle z_\lambda(\cdot), \breve{w}\rangle\, d\mu \to \int \langle z(\cdot), \breve{w}\rangle\, d\mu$ if $\breve{w} \in Z^*$ and $\breve{w}$ is a function for which $z_i^* = 0$ for i greater than some integer i'. Given $w = \sum_{i=1}^\infty z_i^* \chi_{A_i} \in Z^*$, and let $W = \|w\|_\infty$. Let ϵ be an arbitrary strictly positive number. If $B_i = \bigcup_{j>i} A_j$, for i big enough, $\int_{B_i} \|h\| < \epsilon/3W$, for all $h \in \{z\} \cup H$, by equi-integrability. Below, let i be such an integer.

Now, there exists a λ' such that

$$\left| \int \langle z_\lambda(\cdot), \breve{w}\rangle\, d\mu - \int \langle z(\cdot), \breve{w}\rangle\, d\mu \right| < \epsilon/3 \quad \text{for } \lambda \geq \lambda',$$

where $\breve{w} = \sum_{j=1}^i z_j^* \chi_{A_j}$. Thus, for $\lambda \geq \lambda'$, $|\int \langle z_\lambda(\cdot), w\rangle\, d\mu - \int \langle z(\cdot), w\rangle\, d\mu| < \epsilon$, since $|\int \langle h(\cdot), w - \breve{w}\rangle\, d\mu| < \epsilon/3$, for $h \in \{z\} \cup H$. Finally, since Z^* is norm-dense in $L_\infty(\Omega, X^*)$, and $\{z_\lambda(\cdot)\}$ is bounded, we obtain that $\int \langle z_\lambda(\cdot), w\rangle\, d\mu \to \int \langle z(\cdot), w\rangle\, d\mu$ for all $w \in L_\infty(\Omega, X^*)$.

Finally, let us prove the theorem for $p = \infty$. Let $z_\lambda(\cdot) \in L_\infty(\Omega, X)$. We may consider $z_\lambda(\cdot)$ to be a net in $L_2(\Omega, X)$, obviously bounded. As a net in $L_2(\Omega, X)$, it has a cluster point z in this space with respect to the pairing topology. From the proof of Theorem 1 we see that the limit $z(\cdot)$ is essentially bounded when the net $\{z_\lambda(\cdot)\}$ is norm-bounded, since $x(\cdot, 0)$ obviously is essentially bounded when all $y_n(\cdot, 0)$ are uniformly essentially bounded.

If h^N is the indicator function of the set $\{t: \|h(t)\| > N\}$ for a given $h \in L_1(\Omega, X^*)$, we know that $\int_\Omega h^N \|h\| \to 0$, when $N \to \infty$. Thus also, if w is equal to z or any of the z_λ's, $\int h^N \|h\| \cdot \|w\| \to 0$, when $N \to \infty$, this uniformly in λ. Let $h_1^N = 1 - h^N$. Then

$$\int \langle z(\cdot), h\rangle \, d\mu = \lim_N \int \langle z(\cdot), h_1^N h\rangle \, d\mu = \lim_N \left(\lim_\lambda \int \langle z_\lambda(\cdot), h_1^N h\rangle \, d\mu\right)$$

$$= \lim_\lambda \left(\lim_N \int \langle z_\lambda(\cdot), h_1^N h\rangle \, d\mu\right) = \lim_\lambda \int \langle z_\lambda(\cdot), h\rangle \, d\mu,$$

and the proof is complete.

Theorem 2. *Omit the condition in Theorem* 1 *that* X *is separable. Then* H *is relatively sequentially compact in the pairing topology.*

Proof. Let $\{z^m(\cdot)\}$ be a sequence from H. We can assume that the functions $z^m(\cdot)$ take values in a fixed closed separable subspace X_0 of X. The proof of Theorem 1 can now be applied to X_0 and X_0^*, with $\{z^m(\cdot)\} = \{z_\lambda(\cdot)\}$. We add the observation that if $\{x_\lambda\}$ is a sequence, it is possible to pick out a convergent *subsequence* of $\{x_\lambda\}$, which makes possible a proof of sequential compactness for the pairing of $L_p(\Omega, X_0)$, $L_q(\Omega, X_0^*)$. The proof is complete by noting if $z^*(\cdot) \in L_q(\Omega, X^*)$, then the function $z_0^*(\cdot)$ obtained by restricting $z^*(t)$ for each t to X_0, is element of $L_q(\Omega, X_0^*)$.

Remark 1. Because of the trivial character of the measure space N, the product measure space is definable, also if Ω is a general nonfinite measure space. For $p \in \langle 1, \infty\rangle$, it is then possible to prove Theorem 1 for Ω nonfinite. For $p = \infty$, or $p = 1$, the result may be extended to the case where Ω is σ-finite (for $p = 1$, the equi-integrability condition has to be strengthened to uniform countable additivity of the set function $E \to \int \|h(\cdot)\| \chi_E \, d\mu$, $h \in H$, in the sense of [1]).

Finally, in Theorem 2, we can assume without loss of generality, that Ω is σ-finite, hence Theorem 2 holds for $p \in [1, \infty]$ (with the uniform countable additivity condition for $p = 1$).

Remark 2. In $L_q(\Omega, X^*)$, compactness, or countable compactness, of subsets H can be obtained with the aid of representation theorems of continuous linear functionals on $L_p(\Omega, X)$. (For $p < \infty$, $(L_p(\Omega, X))^* = L_q(\Omega, X^*)$, if X^* is separable, or X is reflexive. If the latter properties do not hold, scalarwise integrable X^*-valued functions have to be used to represent continuous linear functionals. See [3].) When X is reflexive and $p \in \langle 1, \infty\rangle$, then $L_q(\Omega, X^*) = (L_p(\Omega, X))^*$, $(L_q(\Omega, X^*))^* = L_p(\Omega, X)$ and the closed unit ball in $L_p(\Omega, X)$ is weakly compact. This well-known fact also follows from our proof: It suffices for compactness of unit balls in "paired" spaces to have countable compactness of these closed unit balls in the pairing topologies, and Theorem 2 implies this countable compactness of the closed unit balls.

LITERATURE

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