

## SOME CHARACTERIZATIONS OF REFLEXIVITY

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**ABSTRACT.** The results of R. C. James on characterizations of reflexivity of Banach spaces with an unconditional basis in terms of  $c_0$  and  $l^1$  are extended to arbitrary Banach spaces. Some consequences are obtained.

R. C. James has proved [4, Theorem 2] that a Banach space  $E$  with an unconditional basis is reflexive if and only if  $E$  contains no subspace isomorphic to  $c_0$  or  $l^1$ . This result has been shown to remain valid for any subspace  $E$  of a space with an unconditional basis (of any power) by C. Bessaga and A. Pełczyński [1], [2], who have also proved [2] that such a space  $E$  is reflexive if and only if  $E^*$  contains no subspace isomorphic to  $l^1$ .

The above conditions are clearly necessary for the reflexivity of any Banach space  $E$ . In the present note we shall show that one can add a certain necessary condition to them, which is also satisfied by any (not necessarily reflexive) subspace of a space with an unconditional basis (of any power) in such a way that these conditions together will be also sufficient for reflexivity. Thus, we shall obtain extensions of the above results to characterizations of reflexivity of an arbitrary Banach space  $E$ .

Following A. Pełczyński [6], a Banach space  $E$  is said to have *property (u)*, if for every weak Cauchy sequence  $\{x_n\} \subset E$  there exists a sequence  $\{y_n\} \subset E$  such that (a) the series  $\sum_{i=1}^{\infty} y_i$  is weakly unconditionally Cauchy and (b) the sequence  $\{x_n - \sum_{i=1}^n y_i\}$  converges weakly to 0.

**Theorem.** *For a Banach space  $E$  the following statements are equivalent:*

- 1°.  $E$  is reflexive.
- 2°.  $E$  has property (u) and  $E$  contains no subspace isomorphic to  $c_0$  or  $l^1$ .
- 3°.  $E$  has property (u) and  $E^*$  contains no subspace isomorphic to  $l^1$ .

**Proof.** Assume 1°. Then, by [6, Proposition 2],  $E$  has property (u). Also,  $E^*$  is reflexive and hence  $E^*$  contains no subspace isomorphic to  $l^1$ . Thus,  $1^\circ \Rightarrow 3^\circ$ .

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Assume now  $3^\circ$ . If  $E$  contains a subspace  $G$  isomorphic to  $c_0$ , then  $E^*$  has a quotient space  $E^*/G^\perp$  isomorphic to  $l^1$  and hence (see e.g. [3, p. 63, exercise 2])  $E^*$  has a complemented subspace isomorphic to  $l^1$ , in contradiction with  $3^\circ$ . On the other hand, if  $E$  contains a subspace isomorphic to  $l^1$ , then so does  $E^*$  [7, Proposition 3.3], in contradiction with  $3^\circ$ . Thus,  $3^\circ \Rightarrow 2^\circ$ .

Assume now  $2^\circ$ . Then, since  $E$  has property (u) and contains no subspace isomorphic to  $c_0$ , by [6, Theorem 1] (for a proof see [9, p. 450]),  $E$  is weakly complete. Hence, since  $E$  contains no subspace isomorphic to  $l^1$ , from [8, Corollary 1] it follows that  $E$  is reflexive. Thus,  $2^\circ \Rightarrow 1^\circ$ .

Since every Banach space  $E$  with an unconditional basis and every subspace of such a space have property (u) (by [6, Theorem 3 and Corollary 1]; for a proof see [9, pp. 445–449]), from the above Theorem we obtain, in particular, the results of R. C. James [4] and C. Bessaga and A. Pełczyński [1], [2] mentioned in the introduction.

**Corollary 1.** *A separable Banach space  $E$  is reflexive if and only if (i)  $E$  has property (u) and (ii)  $E^{**}$  is separable.*

**Proof.** Clearly, (i) and (ii) imply  $3^\circ$  of the above Theorem.

**Remark 1.** Combining [8] with [6, Corollary 5], it follows that if a Banach space  $E$  has property (u) and contains no subspace isomorphic to  $l^1$ , then  $E^*$  is weakly complete. This result yields other proofs of Corollary 1 and the implication  $3^\circ \Rightarrow 1^\circ$  of the Theorem.

**Corollary 2.** *The following two conjectures are equivalent:*

$1^\circ$  [5, p. 165]. *Every infinite dimensional Banach space contains an infinite dimensional subspace that is either reflexive or is isomorphic to  $c_0$  or  $l^1$ .*

$2^\circ$ . *Every infinite dimensional Banach space contains an infinite dimensional subspace with property (u),*

**Proof.**  $c_0$ ,  $l^1$  and every reflexive space have property (u), so  $1^\circ \Rightarrow 2^\circ$ . Conversely, if  $G \subset E$  has property (u), then by the above Theorem either  $G$  is reflexive or  $G$  contains a subspace isomorphic to  $c_0$  or  $l^1$ . Thus,  $2^\circ \Rightarrow 1^\circ$ .

**Remark 2.** The conjecture of Corollary 2, if substantiated, would have some interesting consequences, e.g., that every (infinite dimensional) second conjugate space  $E^{**}$  contains a reflexive subspace (of infinite dimension)—or, equivalently, that every conjugate Banach space  $E^*$  has a reflexive quotient space. Indeed, if  $E \supset G$  reflexive, then  $E^{**} \supset E \supset G$ ; if  $E \supset c_0$ , then  $E^* \supset l^1 \supset l^1$ ; finally, if  $E \supset l^1$ , then  $E^* \supset (l^\infty)^* \supset l^2$ .

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