ON PERTURBING BASES OF
COMPLEX EXPONENTIALS IN $L^2(-\pi, \pi)$

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ABSTRACT. A sequence of complex exponentials $\{e^{i\lambda_n t}\}$ is said to be
a Riesz basis for $L^2(-\pi, \pi)$ if each function in the space has a unique
representation $f = \sum c_n e^{i\lambda_n t}$, with $A \sum |c_n|^2 \leq \|f\|^2 \leq B \sum |c_n|^2$. It is
known, for example, that if $|\lambda_n - n| \leq L < \frac{1}{4}$ ($-\infty < n < \infty$), then $\{e^{i\lambda_n t}\}$ is
a Riesz basis. In this note we show that not only the orthonormal basis
$\{e^{int}\}$, but any Riesz basis of complex exponentials can be suitably
perturbed.

1. Introduction. Using the terminology in [6], we say that a sequence
of complex exponentials $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ if there
exist positive constants $A$ and $B$ such that each function in the space has
a unique representation $f = \sum c_n e^{i\lambda_n t}$, with

$$A \sum |c_n|^2 \leq \|f\|^2 \leq B \sum |c_n|^2.$$ 

The theory of these so-called nonharmonic Fourier series was initiated
by Paley and Wiener who showed that if each $\lambda_n$ is real and $|\lambda_n - n| \leq L$
$(-\infty < n < \infty)$, then $\{e^{i\lambda_n t}\}$ is a Riesz basis whenever $L$ is sufficiently
small [8, p. 108]. The best possible upper bound for $L$ was established by
M. I. Kadec [6] who showed that $\{e^{i\lambda_n t}\}$ is a Riesz basis whenever $L < \frac{1}{4}$.

In this note we show that not only the orthonormal basis $\{e^{int}\}$, but any
Riesz basis of complex exponentials can be suitably perturbed and remain
a Riesz basis. This result extends work done in [5] and [10].

Theorem 1. Let $\{e^{i\lambda_n t}\}$ be a Riesz basis for $L^2(-\pi, \pi)$. There exists a
constant $\epsilon > 0$ such that $\{e^{i\mu_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ whenever $|\lambda_n - \mu_n| \leq \epsilon$.

If $\{e^{i\lambda_n t}\}$ is merely assumed to be a Schauder basis for $L^2(-\pi, \pi)$, then we have the following weaker result.

Theorem 2. Let $\{e^{i\lambda_n t}\}$ be a Schauder basis for $L^2(-\pi, \pi)$ and suppose
that the $\lambda_n$ lie in a strip parallel to the real axis. Then the set $\{e^{i\mu_n t}\}$ is
also a Schauder basis for $L^2(-\pi, \pi)$ whenever $\Sigma|\lambda_n - \mu_n| < \infty$.

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2. Exact frames. Duffin and Schaeffer [5] have termed a set of functions \( \{ e^{i\lambda_n t} \} \) a frame over the interval \((-\pi, \pi)\) if there exist positive constants \( A \) and \( B \) such that

\[
A \left| \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt \right|^2 \leq B \left| \int_{-\pi}^{\pi} g(t) dt \right|^2
\]

for every function \( g \) belonging to \( L^2(-\pi, \pi) \). A classical theorem of Paley and Wiener [8] shows that an equivalent characterization is that

\[
A \int_{-\infty}^{\infty} |f(x)|^2 dx \leq \sum_{n} |f(\lambda_n)|^2 \leq B \int_{-\infty}^{\infty} |f(x)|^2 dx
\]

for every function \( f \) which is entire of exponential type \( \pi \) and such that \( f(x) \in L^2(-\infty, \infty) \). It is clear from (2) that a frame is a complete set of functions in \( L^2(-\pi, \pi) \).

A frame is said to be exact if it fails to be a frame on the removal of any function in the set. The following Proposition shows that the theory of Riesz bases and the theory of exact frames are equivalent.

**Proposition.** A set of complex exponentials \( \{ e^{i\lambda_n t} \} \) is a Riesz basis for \( L^2(-\pi, \pi) \) if and only if it is an exact frame over the interval \((-\pi, \pi)\).

**Proof.** Suppose first that \( \{ e^{i\lambda_n t} \} \) is a Riesz basis for \( L^2(-\pi, \pi) \). It is well known [3] that the inequality \( \| \sum_n c_n e^{i\lambda_n t} \|^2 \leq B \Sigma |c_n|^2 \) holds for all finite sequences \( \{ c_n \} \) if and only if

\[
\frac{1}{2\pi} \sum_n \left| \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt \right|^2 \leq B \left| \int_{-\pi}^{\pi} g(t) dt \right|^2
\]

for every function \( g \) belonging to \( L^2(-\pi, \pi) \). This establishes the right-hand side of (2). To derive the other half of the frame condition, we define a mapping

\[
T : L^2(-\pi, \pi) \to l^2 \quad \text{by} \quad (Tg)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt.
\]

Then \( T \) is a bounded linear transformation into \( l^2 \), and the left-hand side of (1) shows that \( T \) is in fact onto [3]. It follows from Banach’s Isomorphism Theorem that \( T \) is bounded below, and hence (2) is established. Moreover, since \( T \) is onto \( l^2 \), the set \( \{ e^{i\lambda_n t} \} \) possesses a unique biorthogonal sequence. Thus, for each \( k \), the set \( \{ e^{i\lambda_n t} \}_{n \neq k} \) fails to be complete in \( L^2(-\pi, \pi) \), and therefore \( \{ e^{i\lambda_n t} \} \) is an exact frame.

The fact that every exact frame is also a Riesz basis was established in [5, p. 361].

It was shown in [5, p. 364] that a frame can be suitably perturbed and remain a frame. In view of the above Proposition, Theorem 1 will follow if we can establish a similar result for exact frames. To this end, we will need the following interesting result of Bade and Curtis [2, p. 394].
Lemma. Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ a bounded linear transformation. Suppose that there exist constants $M > 0$ and $0 < \epsilon < 1$ with the following property: For each $y$ in the unit ball of $Y$, there exists an $x$ in $X$ with $\|x\| \leq M$ and $\|Tx - y\| \leq \epsilon$. Then $T$ is onto.

3. Proof of Theorem 1. Let $\{e^{i\lambda_n t}\}$ be a Riesz basis for $L^2(-\pi, \pi)$ and suppose that $|\lambda_n - \mu_n| \leq \epsilon$. Then $\{e^{i\lambda_n t}\}$ satisfies the frame condition, and hence $\{e^{i\mu_n t}\}$ is also a frame if $\epsilon$ is sufficiently small [5, p. 346]. It remains only to show that for $\epsilon$ small enough, $\{e^{i\mu_n t}\}$ is, in fact, an exact frame, and for this it is enough to establish the existence of a sequence biorthogonal to $\{e^{i\mu_n t}\}$. Let us denote by $H$ the Paley-Wiener space of entire functions of exponential type $\pi$ which are square integrable over the real axis. The following inequality was established in [5, p. 345] for every function $f$ belonging to $H$ and satisfying (3):

$$\sum_n |(\lambda_n - \mu_n)|^2 \leq C \sum_n |(\lambda_n)|^2,$$

where $C = (B/A)(e^{\pi \epsilon} - 1)^2$. Inequalities (3) and (4) show that the mapping $Tf = \{\langle f, \mu_n \rangle\}$ defines a bounded linear transformation from $H$ into $l^2$. We complete the proof by showing that $T$ is, in fact, onto $l^2$.

It follows just as in the proof of the Proposition that the linear transformation $f \to \{\langle f, \lambda_n \rangle\}$ maps $H$ onto $l^2$. Banach's Isomorphism Theorem then shows that the unit ball of $l^2$ can be interpolated in a uniformly bounded way; that is, there exists a constant $M$ with the property that for each sequence $y = \{y_n\}$ belonging to $l^2$, with $\|y\| \leq 1$, the function $f$ in $H$ satisfying $\langle f, \lambda_n \rangle = y_n$ also satisfies $\|f\| \leq M$. Thus, given $y = \{y_n\}$ in the unit ball of $l^2$, if we choose $f$ in $H$ such that $\langle f, \lambda_n \rangle = y_n$, then (4) becomes $\|Tf - y\| \leq C\|y\| \leq C$, with $\|f\| \leq M$. Since $C \to 0$ as $\epsilon \to 0$, $\epsilon$ can be chosen small enough so that $C < 1$. An application of the Lemma then shows that $T$ is onto. Thus, $\{e^{i\mu_n t}\}$ has a biorthogonal sequence and, consequently, is an exact frame. □

4. Proof of Theorem 2. Let $\{f_n\}$ denote the sequence of coefficient functionals associated with $\{e^{i\lambda_n t}\}$. Since the numbers $\|e^{i\lambda_n t}\|$ are bounded away from zero, it follows that $\sup\|f_n\| < \infty$ [9]. Now, $\{e^{i\lambda_n t}\}$ is known to be complete whenever $\Sigma|\lambda_n - \mu_n| < \infty$ [1], so by [9, Remark 10.3], it is enough to show that $\Sigma\|e^{i\lambda_n t} - e^{i\mu_n t}\| < \infty$. Let us suppose that $|\text{Im}(\lambda_n)| \leq M$ and set $|\lambda_n - \mu_n| = \epsilon_n$ and $\epsilon = \Sigma \epsilon_n < \infty$. Then

$$\|e^{i\lambda_n t} - e^{i\mu_n t}\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{i\lambda_n t} - e^{i\mu_n t}|^2 dt \leq \frac{e^{2\pi M}}{2\pi} \int_{-\pi}^{\pi} |e^{i(\lambda_n - \mu_n)t} - 1|^2 dt.$$

Expanding $e^{i\delta t}$ in a Taylor series, we get
whenever $|t| \leq \pi$, and hence,

$$\left| e^{i(\lambda_n - \mu_n)t} - 1 \right| = \left| \sum_{k=1}^{\infty} \frac{i^k(\lambda_n - \mu_n)^k}{k!} \right| \leq \sum_{k=1}^{\infty} \frac{e^{|\lambda_n - \mu_n|k}}{k!},$$

whenever $|t| \leq \pi$, and hence,

$$\left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\| \leq e^{\pi M} \sum_{n} \frac{\epsilon^n e^{\pi M}}{k!}.$$  

Summing over $n$ in (5) gives

$$\sum_{n} \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\| \leq e^{\pi M} \sum_{k=1}^{\infty} \frac{\epsilon^n e^{\pi M}}{k!} = e^{\pi M}(e^{\pi \epsilon} - 1) < \infty,$$

and the proof is complete. \(\square\)

**Remark.** If the set \(\{e^{i\lambda_n t}\}\) is merely assumed to be complete in \(L^2(-\pi, \pi)\), then it is not necessarily true that there exists a single \(\epsilon > 0\) such that \(\{e^{i\mu_n t}\}\) is complete whenever \(|\lambda_n - \mu_n| \leq \epsilon\). For the sequence \(\{e^{i\lambda_n t}\}\), with \(\lambda_n = n - \frac{1}{4} (n \geq 1)\), is complete in \(L^2(-\pi, \pi)\) [7, p. 67], but if we set \(F(z) = \Pi_{n=1}^{\infty} (1 - z^2/\mu_n^2)\), where \(|\mu_n - n| \leq L < \frac{1}{4}\), then \(F\) is entire of exponential type \(\pi [3, p. 186]\) and belongs to \(L^2\) on the real axis \([7, p. 49]\). Thus, under these conditions, \(\{e^{i\mu_n t}\}\) is not complete in \(L^2(-\pi, \pi)\). Whether a Schauder basis of complex exponentials \(\{e^{i\lambda_n t}\}\) can be so perturbed (with a fixed \(\epsilon\)) and remain a Schauder basis appears to be an open question.

**REFERENCES**