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ON PERTURBING BASES OF COMPLEX EXPONENTIALS IN $L^2(-\pi, \pi)$

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ABSTRACT. A sequence of complex exponentials $\{e^{i\lambda_n t}\}$ is said to be a Riesz basis for $L^2(-\pi, \pi)$ if each function in the space has a unique representation $f = \Sigma c_n e^{i\lambda_n t}$, with $A\,\Sigma|c_n|^2 \leq \|f\|^2 \leq B\Sigma|c_n|^2$. It is known, for example, that if $|\lambda_n - n| \leq L < 1/4$ $(-\infty < n < \infty)$, then $\{e^{i\lambda_n t}\}$ is a Riesz basis. In this note we show that not only the orthonormal basis $\{e^{int}\}$, but any Riesz basis of complex exponentials can be suitably perturbed.

1. **Introduction.** Using the terminology in [6], we say that a sequence of complex exponentials $\{e^{i\lambda_n t}\}$ is a *Riesz basis* for $L^2(-\pi, \pi)$ if there exist positive constants A and B such that each function in the space has a unique representation $f = \Sigma c_n e^{i\lambda_n t}$, with

$$A \sum |c_n|^2 \leq \|f\|^2 \leq B \sum |c_n|^2. \tag{1}$$

The theory of these so-called *nonharmonic* Fourier series was initiated by Paley and Wiener who showed that if each λ_n is real and $|\lambda_n - n| \leq L$ $(-\infty < n < \infty)$, then $\{e^{i\lambda_n t}\}$ is a Riesz basis whenever L is sufficiently small [8, p. 108]. The best possible upper bound for L was established by M. I. Kadec [6] who showed that $\{e^{i\lambda_n t}\}$ is a Riesz basis whenever $L < 1/4$.

In this note we show that not only the orthonormal basis $\{e^{int}\}$, but any Riesz basis of complex exponentials can be suitably perturbed and remain a Riesz basis. This result extends work done in [5] and [10].

Theorem 1. *Let $\{e^{i\lambda_n t}\}$ be a Riesz basis for $L^2(-\pi, \pi)$. There exists a constant $\epsilon > 0$ such that $\{e^{i\mu_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ whenever $|\lambda_n - \mu_n| \leq \epsilon$.*

If $\{e^{i\lambda_n t}\}$ is merely assumed to be a Schauder basis for $L^2(-\pi, \pi)$, then we have the following weaker result.

Theorem 2. *Let $\{e^{i\lambda_n t}\}$ be a Schauder basis for $L^2(-\pi, \pi)$ and suppose that the λ_n lie in a strip parallel to the real axis. Then the set $\{e^{i\mu_n t}\}$ is also a Schauder basis for $L^2(-\pi, \pi)$ whenever $\Sigma|\lambda_n - \mu_n| < \infty$.*

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2. **Exact frames.** Duffin and Schaeffer [5] have termed a set of functions $\{e^{i\lambda_n t}\}$ a *frame* over the interval $(-\pi, \pi)$ if there exist positive constants A and B such that

$$A\int_{-\pi}^{\pi}|g(t)|^2\,dt \le \frac{1}{2\pi}\sum_n\left|\int_{-\pi}^{\pi}g(t)e^{i\lambda_n t}\,dt\right|^2 \le B\int_{-\pi}^{\pi}|g(t)|^2\,dt \tag{2}$$

for every function g belonging to $L^2(-\pi, \pi)$. A classical theorem of Paley and Wiener [8] shows that an equivalent characterization is that

$$A\int_{-\infty}^{\infty}|f(x)|^2\,dx \le \sum_n|f(\lambda_n)|^2 \le B\int_{-\infty}^{\infty}|f(x)|^2\,dx \tag{3}$$

for every function f which is entire of exponential type π and such that $f(x) \in L^2(-\infty, \infty)$. It is clear from (2) that a frame is a complete set of functions in $L^2(-\pi, \pi)$.

A frame is said to be *exact* if it fails to be a frame on the removal of any function in the set. The following Proposition shows that the theory of Riesz bases and the theory of exact frames are equivalent.

Proposition. *A set of complex exponentials $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ if and only if it is an exact frame over the interval $(-\pi, \pi)$.*

Proof. Suppose first that $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi, \pi)$. It is well known [3] that the inequality $\|\Sigma c_n e^{i\lambda_n t}\|^2 \le B\Sigma|c_n|^2$ holds for all finite sequences $\{c_n\}$ if and only if

$$\frac{1}{2\pi}\sum_n\left|\int_{-\pi}^{\pi}g(t)e^{i\lambda_n t}\,dt\right|^2 \le B\int_{-\pi}^{\pi}|g(t)|^2\,dt$$

for every function g belonging to $L^2(-\pi, \pi)$. This establishes the right-hand side of (2). To derive the other half of the frame condition, we define a mapping

$$T\colon\ L^2(-\pi, \pi) \to l^2 \quad \text{by} \quad (Tg)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)e^{i\lambda_n t}\,dt.$$

Then T is a bounded linear transformation into l^2, and the left-hand side of (1) shows that T is in fact *onto* [3]. It follows from Banach's Isomorphism Theorem that T is bounded below, and hence (2) is established. Moreover, since T is onto l^2, the set $\{e^{i\lambda_n t}\}$ possesses a unique biorthogonal sequence. Thus, for each k, the set $\{e^{i\lambda_n t}\}_{n\ne k}$ fails to be complete in $L^2(-\pi, \pi)$, and therefore $\{e^{i\lambda_n t}\}$ is an exact frame.

The fact that every exact frame is also a Riesz basis was established in [5, p. 361]. □

It was shown in [5, p. 364] that a frame can be suitably perturbed and remain a frame. In view of the above Proposition, Theorem 1 will follow if we can establish a similar result for exact frames. To this end, we will need the following interesting result of Bade and Curtis [2, p. 394].

Lemma. *Let X and Y be Banach spaces and $T: X \longrightarrow Y$ a bounded linear transformation. Suppose that there exist constants $M > 0$ and $0 < \epsilon < 1$ with the following property: For each y in the unit ball of Y, there exists an x in X with $\|x\| \le M$ and $\|Tx - y\| \le \epsilon$. Then T is onto.*

3. **Proof of Theorem 1.** Let $\{e^{i\lambda_n t}\}$ be a Riesz basis for $L^2(-\pi, \pi)$ and suppose that $|\lambda_n - \mu_n| \le \epsilon$. Then $\{e^{i\lambda_n t}\}$ satisfies the frame condition, and hence $\{e^{i\mu_n t}\}$ is also a frame if ϵ is sufficiently small [5, p. 346]. It remains only to show that for ϵ small enough, $\{e^{i\mu_n t}\}$ is, in fact, an exact frame,and for this it is enough to establish the existence of a sequence biorthogonal to $\{e^{i\mu_n t}\}$. Let us denote by H the Paley-Wiener space of entire functions of exponential type π which are square integrable over the real axis. The following inequality was established in [5, p. 345] for every function f belonging to H and satisfying (3):

$$\sum_n |f(\lambda_n) - f(\mu_n)|^2 \le C \sum_n |f(\lambda_n)|^2, \tag{4}$$

where $C = (B/A)(e^{\pi\epsilon} - 1)^2$. Inequalities (3) and (4) show that the mapping $Tf = \{f(\mu_n)\}$ defines a bounded linear transformation from H into l^2. We complete the proof by showing that T is, in fact, *onto* l^2.

It follows just as in the proof of the Proposition that the linear transformation $f \longrightarrow \{f(\lambda_n)\}$ maps H onto l^2. Banach's Isomorphism Theorem then shows that the unit ball of l^2 can be interpolated in a uniformly bounded way; that is, there exists a constant M with the property that for each sequence $y = \{y_n\}$ belonging to l^2, with $\|y\| \le 1$, the function f in H satisfying $f(\lambda_n) = y_n$ also satisfies $\|f\| \le M$. Thus, given $y = \{y_n\}$ in the unit ball of l^2, if we choose f in H such that $f(\lambda_n) = y_n$, then (4) becomes $\|Tf - y\|^2 \le C\|y\|^2 \le C$, with $\|f\| \le M$. Since $C \longrightarrow 0$ as $\epsilon \longrightarrow 0$, ϵ can be chosen small enough so that $C < 1$. An application of the Lemma then shows that T is onto. Thus, $\{e^{i\mu_n t}\}$ has a biorthogonal sequence and,consequently, is an exact frame. □

4. **Proof of Theorem 2.** Let $\{f_n\}$ denote the sequence of coefficient functionals associated with $\{e^{i\lambda_n t}\}$. Since the numbers $\|e^{i\lambda_n t}\|$ are bounded away from zero, it follows that $\sup\|f_n\| < \infty$ [9]. Now, $\{e^{i\lambda_n t}\}$ is known to be complete whenever $\Sigma|\lambda_n - \mu_n| < \infty$ [1], so by [9, Remark 10.3], it is enough to show that $\Sigma\|e^{i\lambda_n t} - e^{i\mu_n t}\| < \infty$. Let us suppose that $|\mathrm{Im}(\lambda_n)| \le M$ and set $|\lambda_n - \mu_n| = \epsilon_n$ and $\epsilon = \Sigma\epsilon_n < \infty$. Then

$$\|e^{i\lambda_n t} - e^{i\mu_n t}\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{i\lambda_n t} - e^{i\mu_n t}|^2\, dt \le \frac{e^{2\pi M}}{2\pi}\int_{-\pi}^{\pi} |e^{i(\lambda_n - \mu_n)t} - 1|^2\, dt.$$

Expanding $e^{i\delta t}$ in a Taylor series, we get

$$|e^{i(\lambda_n-\mu_n)t}-1| = \left|\sum_{k=1}^{\infty}\frac{i^k(\lambda_n-\mu_n)^k t^k}{k!}\right| \le \sum_{k=1}^{\infty}\frac{\epsilon_n^k\pi^k}{k!}$$

whenever $|t| \le \pi$, and hence,

$$\|e^{i\lambda_n t}-e^{i\mu_n t}\| \le e^{\pi M}\sum_{k=1}^{\infty}\frac{\epsilon_n^k\pi^k}{k!}. \tag{5}$$

Summing over n in (5) gives

$$\sum_n \|e^{i\lambda_n t}-e^{i\mu_n t}\| \le e^{\pi M}\sum_{k=1}^{\infty}\sum_n \frac{\epsilon_n^k\pi^k}{k!} \le e^{\pi M}\sum_{k=1}^{\infty}\frac{\epsilon^k\pi^k}{k!} = e^{\pi M}(e^{\pi\epsilon}-1) < \infty,$$

and the proof is complete. □

Remark. If the set $\{e^{i\lambda_n t}\}$ is merely assumed to be complete in $L^2(-\pi,\pi)$, then it is not necessarily true that there exists a single $\epsilon > 0$ such that $\{e^{i\mu_n t}\}$ is complete whenever $|\lambda_n - \mu_n| \le \epsilon$. For the sequence $\{e^{\pm i\lambda_n t}\}$, with $\lambda_n = n - \frac{1}{4}$ $(n \ge 1)$, is complete in $L^2(-\pi,\pi)$ [7, p. 67], but if we set $F(z) = \prod_{n=1}^{\infty}(1 - z^2/\mu_n^2)$, where $|\mu_n - n| \le L < \frac{1}{4}$, then F is entire of exponential type π [3, p. 186] and belongs to L^2 on the real axis [7, p. 49]. Thus, under these conditions, $\{e^{i\mu_n t}\}$ is *not* complete in $L^2(-\pi,\pi)$. Whether a Schauder basis of complex exponentials $\{e^{i\lambda_n t}\}$ can be so perturbed (with a fixed ϵ) and remain a Schauder basis appears to be an open question.

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