UNIQUENESS FOR NONLINEAR CAUCHY PROBLEMS
IN BANACH SPACES

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ABSTRACT. Recently Medeiros and Diaz and Weinacht have considered
the question of uniqueness for the Cauchy problem for ordinary differential
equations in a complex Hilbert space. The present paper extends their re-
sults to the case of equations in an arbitrary real or complex Banach space.

1. Introduction. We shall establish uniqueness criteria for the Cauchy
problem

\[(d/dt)u(t) = f(t, u(t)),\]

\[u(0) = u_0;\]

here \(f\) takes values in a real or complex Banach space \(X\) and \(u_0 \in X\). Our
results generalize similar results recently obtained by L. A. Medeiros [7] and
J. B. Diaz and R. J. Weinacht [1], who dealt with the case when \(X\) is a
complex Hilbert space. Our proof is even infinitesimally simpler than those
of the above authors; the key tool is an elementary observation of T. Kato
concerning duality maps.

2. Preliminaries. Throughout this paper \(X\) is a real or complex Banach
space with norm \(\|\cdot\|\), and \(u\) is an \(X\)-valued function on an interval \(I\) of real
numbers. We use the terminology of Hille and Phillips [5]. \(u\) is strongly
absolutely continuous iff for each compact subinterval \([a, b]\) of \(I\) and for
each \(\epsilon > 0\), there is a \(\delta > 0\) such that \(\sum_{i=1}^{n} \|u(b_i) - u(a_i)\| < \epsilon\)
whenever \(a \leq a_i < b_i \leq b, i = 1, \ldots, n,\) and \(\sum_{i=1}^{n} (b_i - a_i) < \delta.\) If \(u\) is strongly absolutely
continuous, then the real-valued function \(\|u(\cdot)\|\) is absolutely continuous.

We state here some elementary sufficient conditions for strong absolute
continuity. If \(u\) is strongly differentiable a.e. with strong derivative \(u'\),
and if \(u(t) - u(a) = \int_{a}^{t} u'(s) \, ds\) (Bochner integral) for \(a, t \in I,\) then \(u\)
is strongly absolutely continuous. (The converse, while false in general, is
true if \(X\) is reflexive.) If \(u\) has a weakly continuous strong derivative
(everywhere), then \( u(t) - u(a) = \int_a^t u'(s) \, ds \), and \( u \) is strongly absolutely continuous.

Let \( X^* \) be the antidual of \( X \), i.e., the space of all continuous antilinear (or conjugate linear) functionals on \( X \). The image of \( x \in X \) under \( \phi \in X^* \) will be denoted by \( \langle x, \phi \rangle \). For each \( x \in X \) let \( f(x) \) be the (nonempty) set of all \( \phi \in X^* \) for which \( \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2 \). A duality map of \( X \) is a function \( j: X \to X^* \) such that \( j(x) \in f(x) \) for each \( x \in X \).

We shall rely on the following simple but useful observation of T. Kato [6, p. 510].

**Lemma.** Suppose \( u \) has a weak derivative \( u'(s) \in X \) at \( t = s \), and suppose \( \|u(s)\| \) is differentiable at \( t = s \). Then
\[
\|u(s)\|\frac{d}{dt}\|u(t)\|_{t=s} = \text{Re} \langle u'(s), \phi \rangle
\]
for each \( \phi \in f(u(s)) \).

For completeness we quote Kato's short proof. Since
\[
\text{Re} \langle u(t), \phi \rangle \leq \|u(t)\|\|\phi\| = \|u(t)\|\|u(s)\|
\]
and
\[
\text{Re} \langle u(s), \phi \rangle = \|u(s)\|^2,
\]
we have
\[
\text{Re} \langle u(t) - u(s), \phi \rangle \leq \|u(s)\|(\|u(t)\| - \|u(s)\|).
\]
Dividing both sides by \( t - s \) and letting \( t \to s \) from above and from below, we obtain
\[
\text{Re} \langle u'(s), \phi \rangle \leq \|u(s)\|\frac{d}{dt}\|u(t)\|_{t=s}.
\]

3. The uniqueness theorems.

**Theorem 1.** Let \( f \) have domain \( D \) contained in \([0, c) \times X \) and range contained in \( X \). Suppose that there is a duality map \( j \) of \( X \) such that
\[
(3) \quad \text{Re} \langle f(t, x) - f(t, y), j(x - y) \rangle \leq t^{-1}\|x - y\|^2
\]
whenever \( t > 0 \) and \((t, x), (t, y) \in D \). Then there is at most one solution \( u \) of (1), (2) on \([0, c) \) in the following sense:

(i) \( u \) is strongly absolutely continuous and has a strong right derivative \( u' \) a.e.;

(ii) \( u'(t) = f(t, u(t)) \) a.e. in \([0, c) \);

(iii) \( u(0) = u_0 \);

(iv) \( \text{strong lim}_{t \to 0^+} t^{-1}(u(t) - u(0)) = u_1 \).

(ii) and (iv) together form a slightly weaker condition than the condition
(ii') \( u'(t) = f(t, u(t)) \) for \( t \in [0, c) \backslash N \), where \( N \) is a Lebesgue null set in \((0, c)\).

When \( X \) is a (real or complex) Hilbert space, \( \langle \cdot, \cdot \rangle \) is the inner product on \( X \) and \( j \) is the identity operator; thus the above result generalizes the corresponding results in [7], [1].

Proof of Theorem 1. The basic outline of the proof is as in [7], [1]. Let \( u, v \) be solutions of (1), (2) in the sense of (i)-(iv), with the same \( u_0', u_1' \).

We must show \( u(t) = v(t), 0 \leq t < c \). By Kato’s lemma and (3) we have for a.e. \( t \in [0, c) \),

\[
\|u(t) - v(t)\|(d/dt)\|u(t) - v(t)\| = \text{Re} \langle f(t, u(t)) - f(t, v(t)), j(u(t) - v(t)) \rangle
\]

\[\leq t^{-1}\|u(t) - v(t)\|^2.\]

Claim. \( (d/dt)\|u(t) - v(t)\| \leq t^{-1}\|u(t) - v(t)\| \) a.e. in \([0, c)\).

This is immediate for a.e. \( t \) for which \( u(t) \neq v(t) \). For a.e. \( t \in [0, c) \) for which \( u(t) = v(t) \) we have \( u'(t) = v'(t) \) by (ii), and

\[
(d/dt)\|u(t) - v(t)\| = \lim_{h \to 0+} h^{-1}\|u(t + h) - v(t + h)\|
\]

\[= \lim_{h \to 0+} h^{-1}(u(t) - v(t)) = 0,
\]

and thus our Claim is proved.

From the Claim it follows that \( (d/dt)(t^{-1}\|u(t) - v(t)\|) \leq 0 \) a.e., and so the function \( t \to t^{-1}\|u(t) - v(t)\| \) is monotone nonincreasing on \((0, c)\). Also,

\[
\lim_{t \to 0+} t^{-1}(u(t) - v(t)) = \lim_{t \to 0+} \{t^{-1}(u(t) - u(0)) - t^{-1}(v(t) - v(0))\}
\]

\[= u_1 - u_1 = 0.
\]

It follows easily that \( u(t) = v(t) \) for all \( t \in [0, c) \), and the proof is complete.

We note, following [1], that \( t^{-1} \) in (3) cannot be replaced by \( Mt^{-1} \) for some \( M > 1 \). (Cf. Perron [8] for a counterexample.)

The final result is a generalized Osgood type uniqueness theorem. Let \( \phi(\cdot, \cdot) \) be a nonnegative measurable function defined on \((0, c) \times [0, \infty)\). Call \( \phi \) a permissible function iff whenever \( w \) is a nonnegative solution of \( w(t) \leq \int^t_0 \phi(s, w(s)) \text{ ds} \) a.e. in \([0, c)\), then \( w = 0 \) a.e.

Theorem 2. Let \( f \) have domain \( D \) contained in \([0, c) \times X \) and range contained in \( X \). Let \( \phi \) be a permissible function, and suppose there is a duality map \( j \) of \( X \) such that

\[
\text{Re} \langle f(t, x) - f(t, y), j(x - y) \rangle \leq \|x - y\|\phi(t, \|x - y\|)
\]

whenever \( t > 0 \) and \((t, x), (t, y) \in D \). Then there is at most one solution \( u \) of (1), (2) satisfying (i)-(iv) of Theorem 1.
Theorem 1 is the special case of Theorem 2 which corresponds to taking \( \phi(t, r) = r/t \). Theorem 2 generalizes results in [7], [1]. For examples of permissible functions see Walter [9, pp. 80–85]. We gratefully acknowledge a suggestion of Hans Weinberger which led to the present formulation of Theorem 2.

**Proof of Theorem 2.** As in the proof of Theorem 1, let \( u, v \) be solutions to (1), (2) satisfying (i)–(iv). Arguing as in that proof we get
\[
\frac{d}{dt} \|u(t) - v(t)\| = \Re \langle f(t, u(t)) - f(t, v(t)), j(u(t) - v(t)) \rangle 
\leq \|u(t) - v(t)\| \phi(t, \|u(t) - v(t)\|);
\]
and we conclude as before that
\[
\frac{d}{dt} \|u(t) - v(t)\| \leq \phi(t, \|u(t) - v(t)\|) \quad \text{a.e. in } [0, c).
\]
Since \( \phi \) is permissible, it follows that \( \|u(t) - v(t)\| = 0 \) a.e., and the proof is complete.

4. Concluding remarks. The map \( j \), defined by
\[
j(f)(x) = \begin{cases} 
0 & \text{if } f(x) = 0; \\
\|f\|^{2-p} |f(x)|^{p-2} & \text{otherwise},
\end{cases}
\]
is a duality map for the real or complex space \( L^p(\Omega) \) on an arbitrary measure space \( \Omega \) for \( 1 \leq p < \infty \). For a brief discussion of duality maps in Orlicz spaces, see [4].

The techniques in this paper can be used to extend other results concerning ordinary differential equations in a complex Hilbert space to an arbitrary Banach space setting. For instance, in Flett’s paper [3], Theorems 2 and 4 (on pp. 336, 341) hold if, in Flett’s notation, \( Y \) is a real or complex Banach space and \( \langle f(t, y) - f(t, z), y - z \rangle \) is replaced by \( \langle f(t, y) - f(t, z), j(y - z) \rangle \), where \( j \) is a duality map of \( Y \).

New uniqueness theorems for the Euler-Poisson-Darboux equation and other singular linear equations can be obtained as an application of the results of this paper. These results will appear elsewhere (cf. Donaldson-Goldstein [2]).

**REFERENCES**


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