REMARKS ON DIFFERENTIAL INEQUALITIES IN BANACH SPACES

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ABSTRACT. Under certain conditions a comparison is made between solutions of a pair of initial value problems in a Banach space. This comparison includes and unifies several recent results on differential inequalities in Banach spaces.

Let $E$ be a real or complex Banach space with the norm on $E$ denoted by $| \cdot |$. Suppose that $T > 0$ and that $A$ and $B$ are continuous functions from $[0, T] \times E$ into $E$. In this note we consider differential inequalities and invariance properties associated with the initial value problems

(1) $u'(t) = A(t, u(t)), \quad u(\lambda) = z, \quad \lambda \leq t \leq T,$

and

(2) $v'(t) = B(t, v(t)), \quad v(\lambda) = w, \quad \lambda \leq t \leq T,$

where $0 \leq \lambda < T$ and $z$ and $w$ are in $E$. The results presented here have considerable overlap with recent works in this area by Redheffer, Volkmann and Walter. See, e.g., [5]–[7] and the results of Bittner [1].

Suppose that $z \in E$, $\delta > 0$ and $\lambda, \mu \in [0, T]$ with $\lambda < \mu$. A continuous function $\alpha = \alpha(\cdot; \lambda, \mu, z, \delta)$ from $[\lambda, \mu]$ into $E$ is said to be an $\delta$-approximate solution to (1) on $[\lambda, \mu]$ if $\alpha(\lambda) = z$ and $\alpha'(t)$ exists with $|\alpha'(t) - A\alpha(t)| \leq \delta$ for all $t \in [\lambda, \mu]$. We also suppose that $\alpha'(t)$ is integrable on $[\lambda, \mu]$ and $\alpha(t) - \alpha(s) = \int_{s}^{t} \alpha'(r)dr$ for all $t, s \in [\lambda, \mu]$. Analogously we define an $\delta$-approximate solution $\beta = \beta(\cdot; \lambda, \mu, w, \delta)$ to (2) on $[\lambda, \mu]$. Now let $K$ be a closed, convex (nonempty) subset of $E$. We write that $A \sim_{K} B$ if whenever $z, w \in E$ with $z - w \in K$ and $\lambda \in [0, T]$, it follows that

(3) $\liminf_{h \to 0^+} d(z - w + h(A(\lambda, z) - B(\lambda, w)); K) / h = 0,$

where $d(y; K) = \inf \{ |y - x| : x \in K \}$ for each $y \in E$.

For our first result we use techniques similar to those of Martin [3] to relate the concept of $A \sim_{K} B$ to the existence of certain types of $\delta$-approximate solutions to (1) and (2).

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Proposition 1. The following statements are equivalent.

(i) \( A \sim K B \).

(ii) If \( z, w \in E \) with \( z - w \in K \), \( \lambda \in [0, T] \), and \( \delta > 0 \), there is a \( \mu = \mu(z, w, \lambda, \delta) \in (\lambda, T] \) and \( \delta \)-approximate solutions \( \alpha = \alpha(\cdot; \lambda, \mu, z, \delta) \) to (1) and \( \beta = \beta(\cdot; \lambda, \mu, w, \delta) \) to (2) such that \( \alpha(t) - \beta(t) \in K \) for all \( t \in [\lambda, \mu] \).

(iii) If \( z, w \in E \) with \( z - w \in K \) and \( \lambda \in [0, T) \), then

\[
\lim_{h \to 0^+} \frac{d(z - w + h(A(\lambda, z) - B(\lambda, w))); K}{h} = 0.
\]

Proof. Suppose first that (i) holds and that \( z, w \in E \) with \( z - w \in K \), \( \lambda \in [0, T) \) and \( \delta > 0 \). Since \( A \sim K B \) there is a \( \delta \in (0, T - \lambda] \) and a \( \mu \in K \) such that

\[
|z + \delta A(\lambda, z) - (w + \delta B(\lambda, w)) - \mu| < \frac{\delta \delta}{2}.
\]

Define \( \mu = \lambda + \delta \) and define \( \beta(t) = w + (t - \lambda)B(\lambda, w) \) and

\[
\alpha(t) = \delta^{-1}[(\mu - t)z + (t - \lambda)(\beta(\mu) + p)] \quad \text{for all } t \in [\lambda, \mu].
\]

Since

\[
\alpha(t) - \beta(t) = \delta^{-1}(\mu - t)(z - w) + \delta^{-1}(t - \lambda)p,
\]

we see that \( \alpha(t) - \beta(t) \in K \) for all \( t \in [\lambda, \mu] \). Noting that \( \beta(0) = w \), \( |\beta(t) - w| \leq \delta|B(\lambda, w)| \) and \( |\beta'(t) - B(t, \beta(t))| = |B(\lambda, w) - B(t, \beta(t))| \), one sees that \( \delta \) can also be chosen so that \( \beta \) is an \( \delta \)-approximate solution to (2) on \( [\lambda, \mu] \). Also \( \alpha(0) = z \) and

\[
|\alpha(t) - z| = \delta^{-1}(t - \lambda)| - z + \beta(\mu) + p| \leq |z - w - \delta B(\lambda, w) - p| \leq \delta \delta/2 + \delta|A(\lambda, z)|,
\]

so it may be assumed that \( |A(t, \alpha(t)) - A(\lambda, z)| \leq \delta \delta/2 \) for all \( t \in [\lambda, \mu] \) as well. Consequently,

\[
|\alpha'(t) - A(t, \alpha(t))| \leq \delta^{-1}|z + w + \delta B(\lambda, w) + p - \delta A(\lambda, z)| + \delta \delta/2 \leq \delta^{-1}\delta \delta/2 + \delta \delta/2 = \delta,
\]

and so \( \alpha \) is an \( \delta \)-approximate solution to (1) on \( [\lambda, \mu] \). This shows that (i) implies (ii). Now suppose that (ii) holds and \( z, w \in E \) with \( z - w \in K \), \( \lambda \in (0, T) \), and \( \delta > 0 \). If \( \mu \in (\lambda, T) \) and \( \alpha = \alpha(\cdot; \lambda, \mu, z, \delta) \) and \( \beta = \beta(\cdot; \lambda, \mu, w, \delta) \) are as in (ii),

\[
\limsup_{h \to 0^+} \frac{d(z - w + h(A(\lambda, z) - B(\lambda, w))); K}{h} \leq \limsup_{h \to 0^+} \frac{|z - w + h(A(\lambda, z) - B(\lambda, w)) - (\alpha(\lambda + h) - \beta(\lambda + h))|}{h}
\]

\[
\leq \limsup_{h \to 0^+} \frac{|A(\lambda, z) - [\alpha(\lambda + h) - z]|}{h} + \frac{|B(\lambda, w) - [\beta(\lambda + h) - w]|}{h} = |A(\lambda, \alpha(\lambda)) - \alpha'(\lambda)| + |B(\lambda, \beta(\lambda)) - \beta'(\lambda)| \leq 2\delta.
\]
Thus (ii) implies (iii). Since the assertion (iii) implies (i) is trivial, the proof is complete.

Now, for convenience, we introduce notations and suppositions which are frequently assumed to hold:

(C1) $K$ is a closed, convex subset of $E$ and $z, w \in E$ with $z - w \in K$.

(C2) $A \sim_K B$.

(C3) There are numbers $M > 1$ and $R > 0$ such that $T \leq R/M$, $|A(t, x)| \leq M - 1$ if $t \in [0, T]$ and $|x - z| \leq R$, and $|B(t, x)| \leq M - 1$ if $t \in [0, T]$ and $|x - w| \leq R$.

Instead of assuming various conditions on $A$ (or $B$) which guarantee the existence of solutions to (1), we use the following criteria: if (C1)–(C3) hold the function $A$ is said to satisfy condition (H) at $z$ if whenever $\{\bar{a}_n\}_{n=1}^\infty$ is a sequence in $(0, \infty)$ with $\lim_{n \to \infty} \bar{a}_n = 0$ and $a_n$ is an $\bar{a}_n$-approximate solution to (1) on $[0, T]$ with $a_n(0) = z$, it follows that $\{a_n\}_{n=1}^\infty$ has a subsequence which converges uniformly to a solution $u$ to (1) on $[0, T]$. An analogous definition applies to the function $B$ satisfying condition (H) at $w$.

Remark 1. There are various conditions that one may place on $A$ in order that it satisfies condition (H) at $z$. For example, one may assume that $A = A_1 + A_2$ where $A_1$ is completely continuous and there is a number $L > 0$ such that $|A_2(t, x) - A_2(t, y)| \leq L|x - y|$ for all $t \in [0, T]$ and all $x, y \in E$ with $|x - z|, |y - z| \leq R$. Further examples involving dissipative and compactness conditions may be found in [4]. It is clear that condition (H) implies the existence of a solution $u$ to (1) on $[0, T]$. Although the author is uncertain, it is probably the case that the existence of a solution to (1) on $[0, T]$ does not imply that condition (H) is satisfied.

Remark 2. Note that if (C3) holds and $\alpha$ is an $\bar{a}$-approximate solution to (1) on $[0, S]$ with $S < T$ and $\bar{a} \in (0, 1]$, then $|\alpha'(t)| \leq |A(t, \alpha(t)) + \bar{a}| \leq M$, and hence $|\alpha(t) - \alpha(s)| \leq M|t - s|$ for all $t, s \in [0, S]$. Similarly, if $\beta$ is an $\bar{a}$-approximate solution to (2) on $[0, S]$ then $|\beta(t) - \beta(s)| \leq M|t - s|$ for all $t, s \in [0, S]$.

Theorem 1. Suppose that (C1)–(C3) are fulfilled, $A$ satisfies condition (H) at $z$, and that $v: [0, T] \to E$ is a solution to (2) on $[0, T]$ with $v(0) = w$. Then there is a solution $u$ to (1) on $[0, T]$ such that $u(0) = z$ and $u(t) - v(t) \in K$ for all $t \in [0, T]$.

Proof. Let $\bar{a} \in (0, 1]$ and let $P_{\bar{a}}$ denote the class of all $\bar{a}$-approximate solutions $\alpha$ to (1) on $[0, S]$ with $0 \leq S \leq T$, $\alpha(S) - v(S) \in K$, and $d(\alpha(t) - v(t); K) \leq \bar{a}$. Partially order $P_{\bar{a}}$ by $\alpha_1 \leq \alpha_2$ only in case $\alpha_2$ is an extension of $\alpha_1$. Using Zorn's lemma (see also Remark 2), one easily sees that $P_{\bar{a}}$ has a maximal element $\alpha$. Suppose, for contradiction, that $\alpha$ is defined on $[0, S]$ with $S < T$. Using (iii) of Proposition 1 and the fact...
that \( \alpha(S) - \nu(S) \in K \), it follows that

\[
\lim_{h \to 0^+} d(\alpha(S) + hA(S, \alpha(S)) - \nu(S + h); K)/h
\]

\[
= \lim_{h \to 0^+} d(\alpha(S) - \nu(S) + h\left[A(S, \alpha(S)) - \int_S^{S+h} B(s, \nu(s)) \, ds\right]; K)/h
\]

\[
\leq \lim_{h \to 0^+} d(\alpha(S) - \nu(S) + h[A(S, \alpha(S)) - B(S, \nu(S))]; K)/h
\]

\[
+ \lim_{h \to 0^+} h^{-1} \int_S^{S+h} |B(s, \nu(s)) - B(S, \nu(S))| \, ds
\]

\[
= 0;
\]

and hence there is a \( \delta > 0 \) and a \( p \in K \) such that \( S + \delta \leq T \) and

\[
|\alpha(S) + \delta A(S, \alpha(S)) - \nu(S + \delta) - p| \leq \delta \bar{e}/2.
\]

Now define the function \( \theta: [0, \delta] \to E \) by

\[
\theta(t) = \delta^{-1}\left[(\delta - t)\alpha(S) + t(\nu(S + \delta) + p)\right].
\]

Then \( \theta(0) = \alpha(S) \) and \( \theta(\delta) - \nu(S + \delta) = p \in K \). Assuming also that \( \delta \) is sufficiently small so that \( |\theta(t) - \alpha(S)| \leq \delta/2 \) and \( |\nu(S + t) - \nu(S)| \leq \delta/2 \) for all \( t \in [0, \delta] \), we have that

\[
d(\theta(t) - \nu(S + t); K) \leq |\theta(t) - \alpha(S)| + |\nu(S) - \nu(S + t)| + d(\alpha(S) - \nu(S); K) \leq \delta/2.
\]

Moreover, if \( t \in [0, \delta] \) then

\[
|A(S, \alpha(S)) - \theta^*_t(t)| = \delta^{-1}|\alpha(S) + \delta A(S, \alpha(S)) - \nu(S + \delta) - p| \leq \delta \bar{e}/2.
\]

In particular, \( |\theta^*_t(t)| \leq M \) and hence \( |\theta(t) - \alpha(S)| \leq Mt \leq M\delta; \) so we may also assume that \( |A(S, \alpha(S)) - A(S + t, \theta(t))| \leq \delta/2 \) for all \( t \in [0, \delta] \). Thus

\[
|\theta^*_t(t) - A(S + t, \theta(t))| \leq \delta \text{ and it follows that if } \alpha_0(t) = \alpha(t) \text{ for } t \in [0, S] \text{ and } \alpha_0(t) = \theta(t - S) \text{ for } t \in [S, S + \delta], \text{ then } \alpha_0 \in P_E \text{ and } \alpha_0 \text{ is a proper extension of } \alpha. \]

This contradiction shows that \( \alpha \) must be defined on \( [0, T] \). In particular, for each \( \delta \in (0, 1) \) there is an \( \delta \)-approximate solution \( \alpha_{\delta} \) to (1) on \( [0, T] \) such that

\[
d(\alpha_{\delta}(t) - \nu(t); K) \leq \delta \bar{e}. \]

Using the fact that \( A \) satisfies condition (H) at \( z \) it now follows immediately that there is a solution \( u \) to (1) on \( [0, T] \) such that \( u(t) - \nu(t) \in K \) for all \( t \in [0, T] \), and the proof is complete.

Remark 3. Note, that with the suppositions of Theorem 1, one can be assured only that there is at least one solution \( u \) to (1) on \( [0, T] \) such that \( u(t) - \nu(t) \in K \) for all \( t \in [0, T] \). As can be shown by very simple examples, it is not necessarily the case that every solution of (1) must satisfy this relationship with \( \nu \).

As a simple example to illustrate Theorem 1, let \( \eta \) be a positive number and take \( K = \{ x \in E : |x| \leq \eta \} \). Let \( F \) be the duality mapping from \( E \) into the class of subsets of the dual space \( E^* \) of \( E \) (i.e., \( Fx = \{ f \in E^* : f(x) = |x| \leq \eta \} \)).
\[ |x|^2 = |f|^2 \] for each \( x \in E \). Then \( A \sim_K B \) only in case \( \text{Re} \ f(A(\lambda, x) - B(\lambda, y)) \leq 0 \) whenever \( \lambda \in [0, 1] \), \( x, y \in E \) with \( |x - y| = \eta \), and \( f \in F(x - y) \).

Of course, the most important examples of convex sets \( K \) are provided by cones. We say that \( K \) is a closed cone in \( E \) if \( K \) is closed and whenever \( x, y \in K \) and \( a \geq 0 \), it follows that \( x + y \in K \) and \( ax \in K \). The above results can be improved somewhat when \( K \) is a cone; so, in place of (C1), we assume the following condition:

\[(C1)' \quad K \text{ is a closed cone in } E \text{ and } z, w \in E \text{ with } z - w \in K. \text{ Moreover, for notational convenience, if } x, y \in E \text{ with } x - y \in K, \text{ we write } x \geq y \text{ or } y \leq x. \text{ In this case we have the following analogous result to Theorem 1:} \]

**Theorem 2.** Suppose that (C1)', (C2) and (C3) are fulfilled, \( A \) satisfies condition (H) at \( z \), and that \( v: [0, T] \to E \) is continuously differentiable with \( v(0) = 0 \) and \( v'(t) \leq B(t, v(t)) \) for all \( t \in [0, T] \). Then there is a solution \( u \) to (1) on \([0, T] \) such that \( u(0) = z \) and \( u(t) \geq v(t) \) for all \( t \in [0, T] \).

**Proof.** Set \( g(t) = B(t, v(t)) - v'(t) \) for all \( t \in [0, T] \) and define

\[ B_1(t, x) = B(t, x) - g(t) \quad \text{for all } (t, x) \in [0, T] \times E. \]

One has that \( v'(t) = B_1(t, v(t)) \) for all \( t \in [0, T] \) and hence Theorem 2 will be a direct consequence of Theorem 1 once it is shown that \( A \sim_K B_1 \).

Note that, since \( K \) is a cone, if \( x \in E \) and \( y \in K \) then \( d(x + y; K) \leq d(x; K) \) (for if \( \{ p_n \}_{n=1}^{\infty} \) is a sequence in \( K \) such that \( d(x; K) = \lim_{n \to \infty} |x - p_n| \), then \( p_n + y \in K \) and hence \( d(x + y; K) \leq \lim_{n \to \infty} |(x + y) - (p_n + y)| = d(x; K) \)).

Since \( B(t, v(t)) \in K \) for all \( b > 0 \) and \( t \in [0, T] \), we see that if \( x \geq y \) and \( t \in [0, T] \),

\[
\lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, y)]; K)}{h} = \lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, y)] + h g(t); K)}{h} \leq \lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, y)]; K)}{h} = 0.
\]

Thus \( A \sim_K B_1 \) and the proof of Theorem 2 is complete.

**Remark 4.** Theorem 2 is not true for general convex subsets \( K \) of \( E \).

As a simple example, take \( K = \{ x \in E: |x| \leq 1 \} \) and define \( A(t, x) = B(t, x) = 0 \) for all \( (t, x) \in [0, 1] \times E \). Set \( z = 0 \) and let \( w \in E \) be such that \( |w| = 1 \). Define \( v(t) = (1 + t)w \) for all \( t \in [0, 1] \). Then \( B(t, v(t)) - v'(t) = -w \in K \) and, since \( u(t) = \theta \) for all \( t \in [0, 1] \), we see that \( u(t) - v(t) = -w \notin K \) and \( v(t) - u(t) = (1 + t)w \notin K \) for any \( t \in (0, 1] \).

In the case that one imposes further smoothness conditions on \( A \) and \( B \), an additional result may be obtained. In the theorem below, we let \( F \) denote the duality mapping from \( E \) into the class of subsets of \( E^* \) (see the paragraph following Remark 3).

**Theorem 3.** Suppose that (C1)' holds, \( A \sim_K A \), \( A \sim_K B \) and \( B \sim_K B \).
Suppose further that there is a number $L > 0$ such that
\[ (4) \Re \left[ f[ A(t, x) - A(t, y)] \right] \leq L |x - y|^2 \quad \text{and} \quad \Re \left[ f[ B(t, x) - B(t, y)] \right] \leq L |x - y|^2 \]
for all $t \in [0, T]$, $x, y \in E$, and $f \in \mathcal{F}(x - y)$. Also, let $u, v : [0, T] \to E$ be continuously differentiable with $u(0) = z$, $v(0) = w$ and
\[ u'(t) - A(t, u(t)) \geq v'(t) - B(t, v(t)) \quad \text{for all } t \in [0, T]. \]
Then $u(t) \geq v(t)$ for all $t \in [0, T]$.

**Indication of proof.** By using a standard argument, it is enough to show that if $0 < \lambda < T$ and $u(t) \geq v(t)$ for all $t \in [0, \lambda]$, then there is a $\delta > 0$ such that $\lambda + \delta < T$ and $u(t) \geq v(t)$ for all $t \in [\lambda, \lambda + \delta]$. So assume that $u(t) \geq v(t)$ for all $t \in [0, \lambda]$ and that $0 \leq \lambda < T$. Define $g_1(t) = v'(t) - B(t, v(t))$ for all $t \in [\lambda, T]$ and $A_1(t, x) = A(t, x) + g_1(t)$ for all $(t, x) \in [\lambda, T] \times E$. Set $K_- = \{ -x : x \in K \}$ and note that if $t \in [\lambda, T]$ and $x, y \in E$ with $x - y \in K_-$, then $y - x \in K$ and
\[
\lim_{h \to 0^+} \frac{d(x - y + h[A_1(t, x) - A_1(t, y)]; K_-)}{h} = \lim_{h \to 0^+} \frac{d(y - x + h[A(t, y) - A(t, x)]; K)}{h} = 0.
\]
Thus $A_1 \sim_{K_-} A_1$. The dissipative condition (4) on $A_1$, and hence on $A_1$, shows that $A_1$ satisfies condition (H) on $[\lambda, \lambda + \delta]$ at $u(\lambda)$ for some $\delta > 0$, and also that the solution to the initial value problem $u'_1(t) = A_1(t, u(t))$, $u_1(\lambda) = u(\lambda)$, $t \in [\lambda, \lambda + \delta]$ is unique. Thus, by Theorem 2 with $K$ replaced by $K_-$, $A = B = A_1$, and $z = w = u(\lambda)$, we see that $u_1(t) - u(t) \in K$ and hence
\[ (5) \quad u(t) \geq u_1(t) \quad \text{for all } t \in [\lambda, \lambda + \delta], \]
where $u'_1(t) = A_1(t, u_1(t))$, $u_1(\lambda) = u(\lambda)$ and $t \in [\lambda, \lambda + \delta]$. Similarly, defining $g_2(t) = u'(t) - A(t, u(t))$ and $B_1(t, x) = B(t, x) + g_2(t)$ for all $t \in [\lambda, \lambda + \delta]$ and all $x \in E$, we see that $B_1 \sim_{K_1} B_1$ and $v'(t) \leq B_1(t, v(t))$ for all $t \in [\lambda, \lambda + \delta]$. Again using (4) and applying Theorem 2 with $A = B = B_1$ and $z = w = v(\lambda)$, we see that
\[ (6) \quad v(t) \leq v_1(t) \quad \text{for all } t \in [\lambda, \lambda + \delta], \]
where $v'_1(t) = B_1(t, v_1(t))$, $v_1(\lambda) = v(\lambda)$ and $t \in [\lambda, \lambda + \delta]$. Now let $x, y \in E$ with $x \geq y$. Noting that $g_1(t) \geq g_2(t)$ we have that $g_1(t) - g_2(t) \in K$ and hence, since $A \sim_{K} B$,
\[
\lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, x)]; K)}{h} = \lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, x)] + h[g_1(t) - g_2(t)]; K)}{h} \\
\leq \lim_{h \to 0^+} \frac{d(x - y + h[A(t, x) - B(t, x)]; K)}{h} = 0.
\]
Thus $A \sim K B$ and Theorem 1 with $A = A_1$ and $B = B_1$ implies since solutions are unique that

\[ u(t) \geq v(t) \quad \text{for all } t \in [\lambda, \lambda + \delta]. \tag{7} \]

Inequalities (5), (6) and (7) show that $u(t) \geq v(t)$ for all $t \in [\lambda, \lambda + \delta]$ and the assertion of Theorem 3 now follows.

**Remark 5.** When $A = B$ in Theorem 3 and the dissipative assumption (4) is replaced by a Lipschitz condition, then we obtain a result of Volkmann [7]. The proof techniques here and those of Volkmann [7] are considerably different, however.

**Remark 6.** The definition of $A \sim K B$ (see equation (3)) can also be expressed in terms of inequalities involving members of the dual space $E^*$ of $E$. If $K$ is a closed convex subset of $E$ and $z, w \in E$ with $z - w \in \partial K$ (where $\partial K$ is the boundary of $K$), then (3) holds only in case $\Re \left( f(A(\lambda, z) - B(\lambda, w)) \right) \leq 0$ whenever $f \in E^*$ with the property that $f(z - w) = r \geq 0$ and $\Re f(x) < r$ for all $x \in K$. See Redheffer and Walter [5]. See also Köthe [2, p. 345].

**Remark 7.** The results of Proposition 1 and Theorem 1 are valid if one assumes only that $K$ is a closed subset of $E$. However, one needs an alteration in (ii) of Proposition 1. Instead of assuming that $\alpha(t) - \beta(t) \in K$ for all $t \in [\lambda, \mu]$, we assume that $\alpha(\mu) - \beta(\mu) \in K$ and for each $t \in [\lambda, \mu]$ there is an $\eta(t) \in [\lambda, t]$ such that $t - \eta(t) \leq \delta$ and $\alpha(\eta(t)) - \beta(\eta(t)) \in K$.

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