A STURM-LIOUVILLE THEOREM FOR SOME ODD MULTIVALUED MAPS

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ABSTRACT. Let $T: H \to 2^H$ be the subdifferential of a real l.s.c. convex function on an infinite dimensional, separable, real Hilbert space $H$. Assuming that $T$ is odd (i.e. $T(-u) = -Tu, \forall u \in H$), $0 \in T(0)$, $(I + T)^{-1}$ is compact and $T(0)$ satisfies a geometrical condition, we prove that $T$ has an infinite sequence $\{\lambda_n\}$ of eigenvalues such that $0 < \lambda_n \to +\infty$.

1. Introduction. Let $H$ be an infinite-dimensional, separable, real Hilbert space with inner-product $(,)$ and norm $\| \cdot \|$, and let $\phi: H \to ]-\infty, +\infty[\)$ be a lower semicontinuous (l.s.c.) convex function such that

\begin{equation}
\phi(u) \geq 0, \quad \forall u \in H, \quad \text{and} \quad \phi(0) = 0.
\end{equation}

Let us consider the multivalued map $T = \partial \phi: H \to 2^H$ defined by

$\partial \phi(u) = \{z \in H | \phi(v) - \phi(u) \geq (z, v - u), \forall v \in H\}.$

We call $T$ the subdifferential of $\phi$. This is a maximal monotone operator (cf. [3], for example).

Let us put $\bar{C} = \{C \subset H | 0 \notin C, \ C \text{ is closed and symmetric}\}.$

Following [7] we define the genus of $C \in \bar{C}$, gen$(C)$, as the smallest integer $N$ such that there exists an odd continuous map $f: C \to \mathbb{R}^N \setminus \{0\}$. By definition $\text{gen}(\emptyset) = 0$ and if there is no such $N$, $\text{gen}(C) = \infty$. We have $\text{gen}(\{x \in \mathbb{R}^N | \|x\| = 1\}) = N$ (cf. [7]).

If $A \subset H \setminus \{0\}$ is symmetric, we put (cf. [1]) $\gamma(A) = \sup\{\text{gen}(C) | C \subset A, \ C \text{ is a compact and symmetric}\}$.

We want to prove the following result:

Theorem 1. Suppose that we have (1.1) and

\begin{equation}
\phi(-u) = \phi(u), \quad \forall u \in H;
\end{equation}

(1.3) The operator $P = (I + T)^{-1}: H \to H$ is compact;

(1.4) For a given $R > 0$ we have $\gamma(\{u \in H | \|u\| = R, \ u \notin T(0)\}) = \infty$.

Then, there exists an infinite sequence $(u_n, \lambda_n) \in H \times \mathbb{R}$, verifying

\begin{equation}
0 \leq \lambda_n \to +\infty, \quad \|u_n\| = R(1 + \lambda_n)^{-1} \to 0, \quad \lambda_n u_n \in Tu_n,
\end{equation}

\[n = 1, 2, \ldots\]
Remark. Condition (1.3) is easily verified if we have

(1.6) The set \( \{ u \in H \mid \|u\| \leq c, \phi(u) \leq c \} \) is compact, \( \forall c > 0 \). Condition (1.4) is trivially verified if \( T(0) \) is bounded and so Theorem 1 extends Theorem 1 of [5] (where \( T(0) = \{ 0 \} \)).

As an application of Theorem 1 let us consider \( H = L^2(\Omega) \), where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with smooth boundary \( \Gamma \), and let \( j: \mathbb{R} \to [-\infty, +\infty] \) be a l. s. c. convex function such that

(1.7) \[ j(r) \geq 0, \quad \forall r \in \mathbb{R}, \quad \text{and} \quad j(0) = 0. \]

We define \( \phi: H \to [-\infty, +\infty] \) by

(1.8) \[ \phi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Gamma} j(u) \, d\gamma & \text{if } u \in H^1_0(\Omega) \text{ and } j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \]

where \( H^1_0(\Omega) = \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma \} \), \( H^m(\Omega) \) is the usual Sobolev space and \( \nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}) \).

It has been proved by H. Brézis (cf. [2]) that \( \phi \) is a l. s. c. convex function which satisfies (1.1) and that

\[ \partial \phi(u) = \begin{cases} -\Delta u + \beta(u) & \text{if } u \in H^2(\Omega) \cap H^1_0(\Omega) \text{ and there exists } f \in L^2(\Omega) \text{ such that } f(x) \in j(u(x)) \text{ a.e. on } \Omega, \\ \emptyset & \text{otherwise}, \end{cases} \]

where \( \beta = \partial j: \mathbb{R} \to 2\mathbb{R} \).

Since the injection \( H^1(\Omega) \hookrightarrow L^2(\Omega) \) is compact, the function \( \phi \) verifies (1.6). Furthermore, if \( T = \partial \phi \), we have \( T(0) = \beta(0) \), and so we have the following consequence of Theorem 1:

**Theorem 2.** Suppose that we have (1.7) and

(1.9) \[ j(-r) = j(r), \quad \forall r \in \mathbb{R}. \]

Then, given \( R > 0 \), there exists an infinite sequence \( (u_n, \lambda_n) \in H \times \mathbb{R} \), verifying (1.5) for \( T = \partial \phi, \phi \) defined by (1.8).

2. Proof of Theorem 1. It is well known that \( P = (I + T)^{-1} \) is a single-valued map with domain \( H \) and range \( D(T) = \{ u \in H \mid Tu \neq \emptyset \} \), and that \( P \) is monotone and contracting (cf. [3], for example).

Furthermore, (1.1) implies that \( 0 \in T(0) \) and so \( P(0) = 0 \).

Let \( G: H \to \mathbb{R} \) be the function defined by

\[ G(u) = (u, Pu) - \frac{1}{2} \| Pu \|^2 - \phi(Pu). \]

The function \( G \) is Fréchet differentiable and \( \nabla G(u) = Pu, \quad \forall u \in H \), where \( \nabla G(u) \) denotes the Fréchet derivative of \( G \) at the point \( u \) (cf. [4], [5]).
Hence $G$ is convex, $G(0) = 0$ and $G(u) \geq 0$, $\forall u \in H$.

If we put $\mu = 1 + \lambda$, $\nu = \mu u$, we have $\lambda u \in T u \iff \nu = \mu P u$.

Since we have $T(0) = \{ u \in H | P u = 0 \} = \{ u \in H | G(u) = 0 \}$, we can write

$$G(u) \neq 0 \iff P u \neq 0,$$

$$\{ u \in S_R | G(u) \neq 0 \} = \{ u \in S_R | u \notin T(0) \},$$

where $S_R = \{ u \in H | \| u \| = R \}$.

Furthermore, the operator $P$ is odd by (1.2) and maximal monotone.

Hence we have, by (1.3)

$$u_n \rightharpoonup u \Rightarrow P u_n \rightharpoonup P u,$$

where $\rightharpoonup$ denotes weak convergence in $H$.

We can now apply theorem $A$ of [1] to operators $A = I$ and $B = P$. We conclude that, if $R > 0$ verifies (1.4), there exists an infinite sequence $(\nu_n, \mu_n) \in H \times R$ such that

$$\nu_n \in S_R, \quad \beta_n = G(\nu_n) > 0, \quad \nu_n = \mu_n P \nu_n, \quad n = 1, 2, \ldots,$$

where $\beta_n = \sup_{C \in \mathcal{C}} \min_{u \in C} G(u)$, $\mathcal{C} = \{ C \subseteq S_R | C$ is compact, symmetric and $\text{gen}(C) \geq n \}$. It is easy to see (by adaptation of the proof of Lemma 4.6 of [6, Chapter VI]) that $\beta_n \downarrow 0$. Furthermore $\mu_n = R^2 (P \nu_n, \nu_n)^{-1} \geq 1$, $n = 1, 2, \ldots$.

Let $\nu_k$ be a subsequence of $\nu_n$ such that $\nu_k \rightharpoonup \nu_0$. We have $G(\nu_0) = \lim_k G(\nu_k) = 0$ and so $P \nu_0 = 0$ by (2.1).

Hence $(P \nu_k, \nu_k) \rightharpoonup (P \nu_0, \nu_0) = 0$ and so $\mu_n \rightharpoonup + \infty$.

This completes the proof of Theorem 1.

REFERENCES


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