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A STURM-LIOUVILLE THEOREM FOR SOME ODD MULTIVALUED MAPS

JOÃO-PAULO DIAS AND JESÚS HERNÁNDEZ

ABSTRACT. Let $T: H \to 2^H$ be the subdifferential of a real l. s. c. convex function on an infinite dimensional, separable, real Hilbert space H. Assuming that T is odd (i.e. $T(-u) = -Tu, \forall u \in H$), $0 \in T(0)$, $(I + T)^{-1}$ is compact and $T(0)$ satisfies a geometrical condition, we prove that T has an infinite sequence $\{\lambda_n\}$ of eigenvalues such that $0 \leq \lambda_n \underset{n}{\to} +\infty$.

1. **Introduction.** Let H be an infinite-dimensional, separable, real Hilbert space with inner-product $(\,,\,)$ and norm $\|\cdot\|$, and let $\phi: H \to]-\infty, +\infty]$ be a lower semicontinuous (l. s. c.) convex function such that

$$\phi(u) \geq 0, \quad \forall u \in H, \quad \text{and} \quad \phi(0) = 0. \tag{1.1}$$

Let us consider the multivalued map $T = \partial\phi: H \to 2^H$ defined by

$$\partial\phi(u) = \{z \in H \mid \phi(v) - \phi(u) \geq (z, v - u), \ \forall v \in H\}.$$

We call T the subdifferential of ϕ. This is a maximal monotone operator (cf. [3], for example).

Let us put $\mathcal{C} = \{C \subset H \mid 0 \notin C, \ C \text{ is closed and symmetric}\}$.

Following [7] we define the genus of $C \in \mathcal{C}$, $\operatorname{gen}(C)$, as the smallest integer N such that there exists an odd continuous map $f: C \to \mathbf{R}^N \setminus \{0\}$. By definition $\operatorname{gen}(\varnothing) = 0$ and if there is no such N, $\operatorname{gen}(C) = \infty$. We have $\operatorname{gen}(\{x \in \mathbf{R}^N \mid \|x\| = 1\}) = N$ (cf. [7]).

If $A \subset H \setminus \{0\}$ is symmetric, we put (cf. [1]) $\gamma(A) = \sup\{\operatorname{gen}(C) \mid C \subset A, \ C$ is a compact and symmetric$\}$.

We want to prove the following result:

Theorem 1. *Suppose that we have* (1.1) *and*

$$\phi(-u) = \phi(u), \quad \forall u \in H; \tag{1.2}$$

(1.3) *The operator* $P = (I + T)^{-1}: H \to H$ *is compact;*

(1.4) *For a given* $R > 0$ *we have* $\gamma(\{u \in H \mid \|u\| = R, \ u \notin T(0)\}) = \infty$.

Then, there exists an infinite sequence $(u_n, \lambda_n) \in H \times \mathbf{R}$, *verifying*

$$0 \leq \lambda_n \underset{n}{\to} +\infty, \quad \|u_n\| = R(1 + \lambda_n)^{-1} \underset{n}{\to} 0, \quad \lambda_n u_n \in Tu_n, \quad n = 1, 2, \ldots. \tag{1.5}$$

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Remark. Condition (1.3) is easily verified if we have

(1.6) The set $\{u \in H \mid \|u\| \leq c,\ \phi(u) \leq c\}$ is compact, $\forall c > 0$. Condition (1.4) is trivially verified if $T(0)$ is bounded and so Theorem 1 extends Theorem 1 of [5] (where $T(0) = \{0\}$).

As an application of Theorem 1 let us consider $H = L^2(\Omega)$, where Ω is a bounded open subset of $\mathbf{R}^N$ with smooth boundary Γ, and let $j: \mathbf{R} \to]-\infty, +\infty]$ be a l. s. c. convex function such that

$$j(r) \geq 0, \quad \forall r \in \mathbf{R}, \quad \text{and} \quad j(0) = 0. \tag{1.7}$$

We define $\phi: H \to]-\infty, +\infty]$ by

$$\phi(u) = \begin{cases} \frac{1}{2} \int_\Omega |\operatorname{grad} u|^2\, dx + \int_\Omega j(u)\, dx & \text{if } u \in H_0^1(\Omega) \text{ and } j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

where $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma\}$, $H^m(\Omega)$ is the usual Sobolev space and $\operatorname{grad} u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$.

It has been proved by H. Brézis (cf. [2]) that ϕ is a l. s. c. convex function which satisfies (1.1) and that

$$\partial\phi(u) = \begin{cases} -\Delta u + \beta(u) & \text{if } u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ and there exists } f \in L^2(\Omega) \\ & \qquad \text{such that } f(x) \in \beta(u(x)) \text{ a.e. on } \Omega, \\ \varnothing & \text{otherwise,} \end{cases}$$

where $\beta = \partial j: \mathbf{R} \to 2^{\mathbf{R}}$.

Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the function ϕ verifies (1.6). Furthermore, if $T = \partial\phi$, we have $T(0) \equiv \beta(0)$, and so we have the following consequence of Theorem 1:

Theorem 2. *Suppose that we have* (1.7) *and*

$$j(-r) = j(r), \quad \forall r \in \mathbf{R}. \tag{1.9}$$

Then, given $R > 0$, *there exists an infinite sequence* $(u_n, \lambda_n) \in H \times \mathbf{R}$, *verifying* (1.5) *for* $T = \partial\phi$, ϕ *defined by* (1.8).

2. **Proof of Theorem 1.** It is well known that $P = (I + T)^{-1}$ is a single-valued map with domain H and range $D(T) = \{u \in H \mid Tu \neq \varnothing\}$, and that P is monotone and contracting (cf. [3], for example).

Furthermore, (1.1) implies that $0 \in T(0)$ and so $P(0) = 0$.

Let $G: H \to \mathbf{R}$ be the function defined by

$$G(u) = (u, Pu) - \tfrac{1}{2}\|Pu\|^2 - \phi(Pu).$$

The function G is Fréchet differentiable and $\nabla G(u) = Pu$, $\forall u \in H$, where $\nabla G(u)$ denotes the Fréchet derivative of G at the point u (cf. [4], [5]).

Hence G is convex, $G(0) = 0$ and $G(u) \geq 0$, $\forall u \in H$.

If we put $\mu = 1 + \lambda$, $v = \mu u$, we have $\lambda u \in Tu \Leftrightarrow v = \mu Pv$.

Since we have $T(0) = \{u \in H | Pu = 0\} = \{u \in H | G(u) = 0\}$, we can write

$$G(u) \neq 0 \Leftrightarrow Pu \neq 0, \tag{2.1}$$

$\{u \in S_R | G(u) \neq 0\} = \{u \in S_R | u \notin T(0)\}$, where $S_R = \{u \in H | \ \|u\| = R\}$.

Furthermore, the operator P is odd by (1.2) and maximal monotone. Hence we have, by (1.3)

$$u_n \underset{n}{\rightharpoonup} u \Rightarrow Pu_n \underset{n}{\rightharpoonup} Pu,$$

where $\rightharpoonup$ denotes weak convergence in H.

We can now apply theorem A of [1] to operators $A \equiv I$ and $B \equiv P$. We conclude that, if $R > 0$ verifies (1.4), there exists an infinite sequence $(v_n, \mu_n) \in H \times \mathbf{R}$ such that

$$v_n \in S_R, \quad \beta_n = G(v_n) > 0, \quad v_n = \mu_n P v_n, \qquad n = 1, 2, \ldots,$$

where $\beta_n = \sup_{C \in \mathcal{C}_n} \min_{u \in C} G(u)$, $\mathcal{C}_n = \{C \subset S_R | C$ is compact, symmetric and $\operatorname{gen}(C) \geq n\}$. It is easy to see (by adaptation of the proof of Lemma 4.6 of [6, Chapter VI]) that $\beta_n \underset{n}{\downarrow} 0$. Furthermore $\mu_n = R^2(Pv_n, v_n)^{-1} \geq 1$, $n = 1, 2, \ldots$.

Let v_k be a subsequence of v_n such that $v_k \underset{k}{\rightharpoonup} v_0$. We have $G(v_0) = \lim_k G(v_k) = 0$ and so $Pv_0 = 0$ by (2.1).

Hence $(Pv_k, v_k) \underset{k}{\rightharpoonup} (Pv_0, v_0) = 0$ and so $\mu_n \underset{n}{\rightharpoonup} +\infty$.

This completes the proof of Theorem 1.

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INSTITUTO DE FISICA E MATEMÁTICA, LISBOA, 4, PORTUGAL

ANALYSE NUMÉRIQUE, UNIVERSITÉ DE PARIS VI, 75-PARIS, 5, FRANCE