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A FIXED POINT THEOREM FOR HYPERSPACES OF λ CONNECTED CONTINUA

CHARLES L. HAGOPIAN

ABSTRACT. Suppose that the hyperspace of compact connected subsets $\mathcal{C}(X)$ of a λ connected continuum X can be ϵ-mapped (for each $\epsilon > 0$) into the plane. We prove that X is either arc-like or circle-like. It follows from this theorem and results of J. T. Rogers, Jr. and J. Segal that $\mathcal{C}(X)$ has the fixed point property.

We call a nondegenerate metric space that is compact and connected a *continuum*. The hyperspace $\mathcal{C}(X)$ of a continuum X is the space of compact connected subsets of X with the Hausdorff metric ρ (i.e., $\rho(A, B) = \text{g.l.b.}\{\epsilon \mid A \subset N_\epsilon(B)$ and $B \subset N_\epsilon(A)$, where $N_\epsilon(A)$ is the ϵ-neighborhood of $A\}$) [6].

A *map* is a continuous single-valued function. A continuum X is said to have the *fixed point property* if for each map f of X into itself, there is a point x of X such that $f(x) = x$. A map f of a continuum X is called an *ϵ-map* if for each point y of $f[X]$, the diameter of $f^{-1}(y)$ is less than ϵ.

A continuum X is *arc-like* if for each positive number ϵ, there is an ϵ-map of X onto an arc. *Circle-like* and *disk-like* continua are defined in the same manner.

A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *hereditarily decomposable* if all of its subcontinua are decomposable. If every pair of points in a continuum X lies in a hereditarily decomposable subcontinuum of X, then X is said to be *λ connected.*

A continuum T is called a *triod* if it contains a subcontinuum Z such that $T - Z$ is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be *atriodic*.

A continuum is *unicoherent* provided that if it is the union of two subcontinua E and F, then $E \cap F$ is connected.

Throughout this paper E^2 is the Euclidean plane and δ is the standard Euclidean metric on E^2. The boundary and the interior of a given set Z are denoted by Bd Z and Int Z respectively.

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We define the following subsets of E^2.

$D_1 = \{(x, y) \mid 0 \leq x \leq 6 \text{ and } 0 \leq y \leq 6\}$, $D_2 = \{(x, y) \mid 0 \leq x \leq 5 \text{ and } 0 \leq y \leq 6\}$,

$D_3 = \{(x, y) \mid 3 \leq x \leq 4 \text{ and } 2 \leq y \leq 4\}$, $D_4 = \{(x, y) \mid 3 \leq x \leq 6 \text{ and } 0 \leq y \leq 6\}$,

$B = \{(x, y) \mid \delta((x, y), (1, 2)) < 1 \text{ or } \delta((x, y), (1, 4)) < 1\}$,

$D = D_1 - B$, and $D' = D_2 - \{(x, y) \mid \delta((x, y), (0, 3)) < 1/3\}$.

Lemma. *There exists a positive number ϵ such that if f is an ϵ-map of D into E^2, then $f[\mathrm{Bd}\, D']$ separates $f((1, 3))$ from $f((6, 3))$ in E^2.*

Proof. Let k be a map of D' onto a 2-sphere S^2 such that (1) $k[\mathrm{Bd}\, D']$ is a point of S^2, (2) k restricted to Int D' is a homeomorphism, and (3) if p and p' are antipodal points of S^2, then $k[D_3] \cap \{p, p'\} \neq \emptyset$. Define ϵ to be a positive number less than ½ such that for each pair p, p' of antipodal points of S^2, $\delta(k^{-1}(p), k^{-1}(p')) > \epsilon$.

Let f be an ϵ-map of D into E^2. Since $f[\mathrm{Bd}\, B]$ and $f[D_4]$ are disjoint continua, there is a map f^* of D_1 into E^2 that is an extension of f such that $f^*[B] \cap f[D_4] = \emptyset$.

The complementary domain G of $f[\mathrm{Bd}\, D']$ that contains $f((1, 3))$ in E^2 is a subset of $f^*[D']$. To see this, assume the contrary. Let w be a point of $G - f^*[D']$. Let d be the projection map of E^2 onto the quotient space E^2/R, where R relates distinct points z and z' of E^2 if and only if $\{z, z'\} \subset E^2 - G$. Since E^2/R is either a 2-sphere or a plane, we can assume that $E^2/R - \{w\}$ is lying in E^2. Note that $h = df^*k^{-1}$ is a map of S^2 into E^2. Also note that since $f[\mathrm{Bd}\, D']$ does not meet the image under f of the line segment in E^2 from (1, 3) to (3, 3), the domain G contains $f[D_3]$. According to the Borsuk-Ulam theorem, there exist antipodal points p and p' of S^2 such that $h(p) = h(p')$. But since f is an ϵ-map and $f^*[B] \cap f[D_3] = \emptyset$, this is a contradiction.

Since $f((6, 3))$ does not belong to $f^*[D']$, it follows that $f((6, 3))$ is not in G. Hence $f[\mathrm{Bd}\, D']$ separates $f((1, 3))$ from $f((6, 3))$ in E^2.

Theorem 1. *Suppose that for each $\epsilon > 0$ the hyperspace $\mathcal{C}(X)$ of a continuum X can be ϵ-mapped into E^2. Then X is atriodic and every proper subcontinuum of X is unicoherent.*

Proof. Assume that X contains a triod. Then $\mathcal{C}(X)$ contains a 3-cell B^3 [9, Theorem 1]. It follows from the Borsuk-Ulam theorem that for some $\epsilon > 0$, Bd B^3 cannot be ϵ-mapped into E^2, which contradicts our hypothesis. Hence X is atriodic.

Suppose that a proper subcontinuum Y of X is not unicoherent. Then there exist continua E_1, E_2 and disjoint nonempty closed sets A_1, A_2 such that $E_1 \cup E_2 = Y$ and $E_1 \cap E_2 = A_1 \cup A_2$. Since X is atriodic, A_1 and A_2 are components of $A_1 \cup A_2$ [7, Theorem 50, p. 18]. We can assume without loss of generality that E_1 and E_2 are both irreducible with respect to intersecting A_1 and A_2.

For $i = 1, 2$ and $j = 1, 2$, let $\mathfrak{A}_{ij}$ be a monotone increasing collection of compact connected subsets of E_j that forms an arc in $\mathcal{C}(X)$ from A_i to E_j. Note that $\bigcup_{i,j} \mathfrak{A}_{ij}$ is a simple closed curve in $\mathcal{C}(X)$.

For $i = 1$ and 2, let h_i be the function of $\mathfrak{A}_{i1} \times \mathfrak{A}_{i2}$ into $\mathcal{C}(X)$ that assigns to each point (X_1, X_2) of $\mathfrak{A}_{i1} \times \mathfrak{A}_{i2}$ the compact connected set $X_1 \cup X_2$. Each h_i is a homeomorphism. The intersection of the disks $h_1[\mathfrak{A}_{11} \times \mathfrak{A}_{12}]$ and $h_2[\mathfrak{A}_{21} \times \mathfrak{A}_{22}]$ is $\{E_1, E_2, Y\}$. In fact, there is a homeomorphism g of $h_1[\mathfrak{A}_{11} \times \mathfrak{A}_{12}] \cup h_2[\mathfrak{A}_{21} \times \mathfrak{A}_{22}]$ onto D (the square with two holes in E^2 defined above) such that $g(A_1) = (0, 3)$, $g(A_2) = (6, 3)$, $g(E_1) = (0, 2)$, $g(E_2) = (0, 4)$, and $g(Y) = (1, 3)$.

According to our Lemma, there is a positive number ϵ_1 such that if f is an ϵ_1-map of $\mathcal{C}(X)$ into E^2, then $f[g^{-1}[\mathrm{Bd}\, D']]$ separates $f(Y)$ from $f(A_2)$ in E^2.

Let $\mathcal{I}$ be a monotone increasing collection of subcontinua of X that forms an arc from Y to X in $\mathcal{C}(X)$. Let u be a point of A_2. If $\{u\} = A_2$, define $\mathcal{J} = \{A_2\}$; otherwise, let $\mathcal{J}$ be a monotone increasing collection of compact connected sets in A_2 that forms an arc in $\mathcal{C}(X)$ from $\{u\}$ to A_2. Define v to be a point of $X - Y$ and let $\mathcal{K}$ be a monotone increasing collection of compact connected sets in X that forms an arc from $\{v\}$ to X in $\mathcal{C}(X)$. Define $\mathcal{S}$ to be the continuum in $\mathcal{C}(X)$ whose points are the singletons of X. Note that $\mathcal{M} = \mathcal{I} \cup \mathcal{J} \cup \mathcal{K} \cup \mathcal{S}$ is a subcontinuum of $\mathcal{C}(X)$ that contains the set $\{Y, A_2\}$ and misses $g^{-1}[\mathrm{Bd}\, D']$.

Define ϵ to be the minimum of ϵ_1 and the distance from $\mathcal{M}$ to $g^{-1}[Bd\, D']$ in $\mathcal{C}(X)$. Let f be an ϵ-map of $\mathcal{C}(X)$ into E^2. It follows that the continuum $f[\mathcal{M}]$ contains $\{f(Y), f(A_2)\}$ and does not intersect $f[g^{-1}[\mathrm{Bd}\, D']]$. But since $f[g^{-1}[\mathrm{Bd}\, D']]$ separates $f(Y)$ from $f(A_2)$ in E^2, this is a contradiction. Hence every proper subcontinuum of X is unicoherent.

Theorem 2. *Suppose that X is a λ connected continuum and that for each $\epsilon > 0$, $\mathcal{C}(X)$ can be ϵ-mapped into E^2. Then X is either arc-like or circle-like.*

Proof. X is atriodic and every proper subcontinuum of X is unicoherent (Theorem 1). If X is unicoherent, then X is hereditarily decomposable [3, Theorem 2] and therefore arc-like [1, Theorem 11]. If X is not unicoherent, then X is circle-like [4, Theorem 2].

Theorem 3. *If X satisfies the hypothesis of Theorem 2, then $\mathcal{C}(X)$ is disk-like and has the fixed point property.*

Proof. If a continuum Y is an arc or a circle, then $\mathcal{C}(Y)$ is a disk. Therefore, since X is arc-like or circle-like, $\mathcal{C}(X)$ is disk-like [12]. Segal [11] proved that for each arc-like continuum Y, the hyperspace $\mathcal{C}(Y)$ has the fixed point property. Rogers [10] showed that $\mathcal{C}(Y)$ has the fixed point property when Y is a circle-like continuum.

Comments. It follows from results in [4] and [5] that true statements are obtained when the phrase "the cone over X" is substituted for "$\mathcal{C}(X)$" in Theorems 2 and 3. In [3] we proved that λ connected continua X and Y are arc-like if and only if $X \times Y$ is disk-like. Hence if X and Y are λ connected continua and $X \times Y$ is disk-like, $X \times Y$ has the fixed point property [2]. We do not have an example of a disk-like continuum that does not have the fixed point property.

$\mathcal{C}(X)$ is embeddable in E^2 if and only if X is an arc or a simple closed curve [8, Theorem 2.3]. For ϵ-mappings we have the following analogue.

Theorem 4. *Suppose that X is a λ connected continuum. Then $\mathcal{C}(X)$ can be ϵ-mapped (for each $\epsilon > 0$) into E^2 if and only if X is arc-like or circle-like.*

Question. Can Theorem 4 be extended to include all continua?

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SACRAMENTO, CALIFORNIA 95819