A FIXED POINT THEOREM FOR HYPERSPACES OF A CONNECTED CONTINUUM

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ABSTRACT. Suppose that the hyperspace of compact connected subsets \( C(X) \) of a \( \lambda \) connected continuum \( X \) can be \( \epsilon \)-mapped (for each \( \epsilon > 0 \)) into the plane. We prove that \( X \) is either arc-like or circle-like. It follows from this theorem and results of J. T. Rogers, Jr. and J. Segal that \( C(X) \) has the fixed point property.

We call a nondegenerate metric space that is compact and connected a continuum. The hyperspace \( \mathcal{C}(X) \) of a continuum \( X \) is the space of compact connected subsets of \( X \) with the Hausdorff metric \( p \) (i.e., \( p(A, B) = \text{g.l.b.} |A \cap N_\epsilon(B) \cap B \cap N_\epsilon(A)| \) for \( A \subseteq N_\epsilon(B) \) and \( B \subseteq N_\epsilon(A) \), where \( N_\epsilon(A) \) is the \( \epsilon \)-neighborhood of \( A \)) [6].

A map is a continuous single-valued function. A continuum \( X \) is said to have the fixed point property if for each map \( f \) of \( X \) into itself, there is a point \( x \) of \( X \) such that \( f(x) = x \). A map \( f \) of a continuum \( X \) is called an \( \epsilon \)-map if for each point \( y \) of \( f[X] \), the diameter of \( f^{-1}(y) \) is less than \( \epsilon \).

A continuum \( X \) is arc-like if for each positive number \( \epsilon \), there is an \( \epsilon \)-map of \( X \) onto an arc. Circle-like and disk-like continua are defined in the same manner.

A continuum is decomposable if it is the union of two proper subcontinua. A continuum is hereditarily decomposable if all of its subcontinua are decomposable. If every pair of points in a continuum \( X \) lies in a hereditarily decomposable subcontinuum of \( X \), then \( X \) is said to be \( \lambda \) connected.

A continuum \( T \) is called a triod if it contains a subcontinuum \( Z \) such that \( T - Z \) is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be atriodic.

A continuum is unicoherent provided that if it is the union of two subcontinua \( E \) and \( F \), then \( E \cap F \) is connected.

Throughout this paper \( E^2 \) is the Euclidean plane and \( \delta \) is the standard Euclidean metric on \( E^2 \). The boundary and the interior of a given set \( Z \) are denoted by \( \text{Bd} Z \) and \( \text{Int} Z \) respectively.

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We define the following subsets of $E^2$.

\[ D_1 = \{(x, y) \mid 0 \leq x \leq 6 \text{ and } 0 \leq y \leq 6\}, \quad D_2 = \{(x, y) \mid 0 \leq x \leq 5 \text{ and } 0 \leq y \leq 6\}, \]

\[ D_3 = \{(x, y) \mid 3 \leq x \leq 4 \text{ and } 2 \leq y \leq 4\}, \quad D_4 = \{(x, y) \mid 3 \leq x \leq 6 \text{ and } 0 \leq y \leq 6\}, \]

\[ B = \{(x, y) \mid \delta((x, y), (1, 2)) < 1 \text{ or } \delta((x, y), (1, 4)) < 1\}, \]

\[ D = D_1 - B, \quad \text{and} \quad D' = D_2 - \{(x, y) \mid \delta((x, y), (0, 3)) < 1/3\}. \]

**Lemma.** There exists a positive number $\epsilon$ such that if $f$ is an $\epsilon$-map of $D$ into $E^2$, then $f[B \cup D']$ separates $f((1, 3))$ from $f((6, 3))$ in $E^2$.

**Proof.** Let $k$ be a map of $D'$ onto a 2-sphere $S^2$ such that (1) $k[B \cup D']$ is a point of $S^2$, (2) $k$ restricted to $\text{Int} D'$ is a homeomorphism, and (3) if $p$ and $p'$ are antipodal points of $S^2$, then $k[D_1] \cap \{p, p'\} \neq \emptyset$. Define $\epsilon$ to be a positive number less than $1/2$ such that for each pair $p, p'$ of antipodal points of $S^2$, $\delta(k^{-1}(p), k^{-1}(p')) > \epsilon$.

Let $f$ be an $\epsilon$-map of $D$ into $E^2$. Since $f[B]$ and $f[D_4]$ are disjoint continua, there is a map $f^*$ of $D_1$ into $E^2$ that is an extension of $f$ such that $f^*[B] \cap f[D_4] = \emptyset$.

The complementary domain $G$ of $f[B \cup D']$ that contains $f((1, 3))$ in $E^2$ is a subset of $f^*[D']$. To see this, assume the contrary. Let $w$ be a point of $G - f^*[D']$. Let $d$ be the projection map of $E^2$ onto the quotient space $E^2/R$, where $R$ relates distinct points $z$ and $z'$ of $E^2$ if and only if $\{z, z'\} \subset E^2 - G$. Since $E^2/R$ is either a 2-sphere or a plane, we can assume that $E^2/R - \{w\}$ is lying in $E^2$. Note that $h = df^*k^{-1}$ is a map of $S^2$ into $E^2$.

Also note that since $f[B \cup D']$ does not meet the image under $f$ of the line segment in $E^2$ from $(1, 3)$ to $(3, 3)$, the domain $G$ contains $f[D_1]$. According to the Borsuk-Ulam theorem, there exist antipodal points $p$ and $p'$ of $S^2$ such that $h(p) = h(p')$. But since $f$ is an $\epsilon$-map and $f^*[B] \cap f[D_4] = \emptyset$, this is a contradiction.

Since $f((6, 3))$ does not belong to $f^*[D']$, it follows that $f((6, 3))$ is not in $G$. Hence $f[B \cup D']$ separates $f((1, 3))$ from $f((6, 3))$ in $E^2$.

**Theorem 1.** Suppose that for each $\epsilon > 0$ the hyperspace $\mathcal{C}(X)$ of a continuum $X$ can be $\epsilon$-mapped into $E^2$. Then $X$ is atriodic and every proper subcontinuum of $X$ is unicoherent.

**Proof.** Assume that $X$ contains a triod. Then $\mathcal{C}(X)$ contains a 3-cell $B^3$ [9, Theorem 1]. It follows from the Borsuk-Ulam theorem that for some $\epsilon > 0$, $\text{Bd} B^3$ cannot be $\epsilon$-mapped into $E^2$, which contradicts our hypothesis. Hence $X$ is atriodic.

Suppose that a proper subcontinuum $Y$ of $X$ is not unicoherent. Then there exist continua $E_1, E_2$ and disjoint nonempty closed sets $A_1, A_2$ such...
that $E_1 \cup E_2 = Y$ and $E_1 \cap E_2 = A_1 \cup A_2$. Since $X$ is atriodic, $A_1$ and $A_2$ are components of $A_1 \cup A_2$ [7, Theorem 50, p. 18]. We can assume without loss of generality that $E_1$ and $E_2$ are both irreducible with respect to intersecting $A_1$ and $A_2$.

For $i = 1, 2$ and $j = 1, 2$, let $\mathcal{A}_{ij}$ be a monotone increasing collection of compact connected subsets of $E_j$ that forms an arc in $\mathcal{C}(X)$ from $A_i$ to $E_j$. Note that $\bigcup_{i,j} \mathcal{A}_{ij}$ is a simple closed curve in $\mathcal{C}(X)$.

For $i = 1$ and 2, let $h_i$ be the function of $\mathcal{A}_{i1} \times \mathcal{A}_{i2}$ into $\mathcal{C}(X)$ that assigns to each point $(X_1, X_2)$ of $\mathcal{A}_{i1} \times \mathcal{A}_{i2}$ the compact connected set $X_1 \cup X_2$. Each $h_i$ is a homeomorphism. The intersection of the disks $h_1[\mathcal{A}_{11} \times \mathcal{A}_{12}]$ and $h_2[\mathcal{A}_{21} \times \mathcal{A}_{22}]$ is $\{E_1, E_2, Y\}$. In fact, there is a homeomorphism $g$ of $h_1[\mathcal{A}_{11} \times \mathcal{A}_{12}] \cup h_2[\mathcal{A}_{21} \times \mathcal{A}_{22}]$ onto $D$ (the square with two holes in $E^2$ defined above) such that $g(A_1) = (0, 3), g(A_2) = (6, 3), g(E_1) = (0, 2), g(E_2) = (0, 4),$ and $g(Y) = (1, 3)$.

According to our Lemma, there is a positive number $\epsilon_1$ such that if $f$ is an $\epsilon_1$-map of $\mathcal{C}(X)$ into $E^2$, then $f[g^{-1}[\partial D']]$ separates $f(Y)$ from $f(A_2)$ in $E^2$.

Let $\mathcal{J}$ be a monotone increasing collection of subcontinua of $X$ that forms an arc from $Y$ to $X$ in $\mathcal{C}(X)$. Let $u$ be a point of $A_2$. If $\{u\} = A_2$, define $\mathcal{J} = \{A_2\}$; otherwise, let $\mathcal{J}$ be a monotone increasing collection of compact connected sets in $A_2$ that forms an arc in $\mathcal{C}(X)$ from $\{u\}$ to $A_2$. Define $\nu$ to be a point of $X - Y$ and let $\mathcal{K}$ be a monotone increasing collection of compact connected sets in $X$ that forms an arc from $\{\nu\}$ to $X$ in $\mathcal{C}(X)$. Define $\mathcal{S}$ to be the continuum in $\mathcal{C}(X)$ whose points are the singletons of $X$. Note that $\mathcal{M} = \mathcal{J} \cup \mathcal{J} \cup \mathcal{K} \cup \mathcal{S}$ is a subcontinuum of $\mathcal{C}(X)$ that contains the set $\{Y, A_2\}$ and misses $g^{-1}[\partial D']$.

Define $\epsilon$ to be the minimum of $\epsilon_1$ and the distance from $\mathcal{M}$ to $g^{-1}[\partial D']$ in $\mathcal{C}(X)$. Let $f$ be an $\epsilon$-map of $\mathcal{C}(X)$ into $E^2$. It follows that the continuum $f[\mathcal{M}]$ contains $f(Y)$, $f(A_2)$, and does not intersect $f[g^{-1}[\partial D']]$. But since $f[g^{-1}[\partial D']]$ separates $f(Y)$ from $f(A_2)$ in $E^2$, this is a contradiction. Hence every proper subcontinuum of $X$ is unicoherent.

**Theorem 2.** Suppose that $X$ is a $\lambda$ connected continuum and that for each $\epsilon > 0$, $\mathcal{C}(X)$ can be $\epsilon$-mapped into $E^2$. Then $X$ is either arc-like or circle-like.

**Proof.** $X$ is atriodic and every proper subcontinuum of $X$ is unicoherent (Theorem 1). If $X$ is unicoherent, then $X$ is hereditarily decomposable [3, Theorem 2] and therefore arc-like [1, Theorem 11]. If $X$ is not unicoherent, then $X$ is circle-like [4, Theorem 2].

**Theorem 3.** If $X$ satisfies the hypothesis of Theorem 2, then $\mathcal{C}(X)$ is disk-like and has the fixed point property.
Proof. If a continuum $Y$ is an arc or a circle, then $\mathcal{C}(Y)$ is a disk. Therefore, since $X$ is arc-like or circle-like, $\mathcal{C}(X)$ is disk-like [12]. Segal [11] proved that for each arc-like continuum $Y$, the hyperspace $\mathcal{C}(Y)$ has the fixed point property. Rogers [10] showed that $\mathcal{C}(Y)$ has the fixed point property when $Y$ is a circle-like continuum.

Comments. It follows from results in [4] and [5] that true statements are obtained when the phrase "the cone over $X$" is substituted for "$\mathcal{C}(X)$" in Theorems 2 and 3. In [3] we proved that $\lambda$ connected continua $X$ and $Y$ are arc-like if and only if $X \times Y$ is disk-like. Hence if $X$ and $Y$ are $\lambda$ connected continua and $X \times Y$ is disk-like, $X \times Y$ has the fixed point property [2]. We do not have an example of a disk-like continuum that does not have the fixed point property.

$\mathcal{C}(X)$ is embeddable in $E^2$ if and only if $X$ is an arc or a simple closed curve [8, Theorem 2.3]. For $\epsilon$-mappings we have the following analogue.

Theorem 4. Suppose that $X$ is a $\lambda$ connected continuum. Then $\mathcal{C}(X)$ can be $\epsilon$-mapped (for each $\epsilon > 0$) into $E^2$ if and only if $X$ is arc-like or circle-like.

Question. Can Theorem 4 be extended to include all continua?

REFERENCES


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