AN ALTERNATE CHARACTERIZATION
OF THE CANTOR SET

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ABSTRACT. Let $X$ be a compact metric space such that, up to homeomorphism, $X$ has only two nonempty open subsets. Then $X$ is homeomorphic to the Cantor discontinuum.

It is well known that any compact, perfect, totally disconnected metric space is homeomorphic to the Cantor "middle thirds" set $K$. The Cantor set is also known to have the following property: Up to homeomorphism, $K$ has only two nonempty open subsets [1]. We will show that, among compact metric spaces, this property characterizes $K$.

Definition. A metric space $X$ has property $W$ if:
(a) $X$ has at least one nonempty compact open subset and at least one noncompact open subset (one of these may be $X$).
(b) Any two nonempty compact open subsets of $X$ are homeomorphic.
(c) Any two noncompact open subsets of $X$ are homeomorphic.

Theorem. Let $X$ be a compact metric space. Then $X$ is homeomorphic to $K$ if and only if $X$ has property $W$.

Proof. The property is preserved by homeomorphism, and is thus necessary. Suppose now that $X$ has property $W$. Any isolated point of $X$ would, by (b), be homeomorphic to $X$, making $X$ a one-point space. This would contradict (a), so $X$ is perfect. We now show that $X$ is disconnected.

Let $x$ and $y$ be distinct points of $X$, $d$ the distance from $x$ to $y$, $U_x$ and $U_y$ the open balls of radius $d/3$ about $x$ and $y$ respectively, and $U = U_x \cup U_y$. If $U$ is homeomorphic to $X$, then $X$ is disconnected. If $U$ is not homeomorphic to $X$, then $X$ is noncompact. Since $X$ is perfect, we have for each $x \in X$ that $X \setminus \{x\}$ is noncompact and open, therefore (being homeomorphic to $U$) disconnected. Thus every point of $X$ is a cut-point of $X$. Here too $X$ must be disconnected, as every metric continuum has at least two non-cut-points.

Finally, take a point $x \in X$ and consider the quasicomponent (equals the component) of $x$, say $C$. The set $C$ cannot be open, for then it would be homeomorphic to $X$ and disconnected. Thus $X \setminus C$ is not compact and is disconnected.

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homeomorphic to $X \setminus \{x\}$. Now each $y \in X \setminus C$ has a compact neighborhood $V \subseteq X \setminus C$ (the complement of a closed-and-open $U \subseteq X$ such that $x \in U$ and $y \notin U$). The same holds for $X \setminus \{x\}$, which shows that $X$ is totally disconnected. As a compact, perfect, totally disconnected metric space, $X$ is homeomorphic to $K$.

**Corollary.** Let $X$ be a noncompact metric space. Then $X$ is homeomorphic to $K \setminus \{0\}$ if and only if $X$ has property $W$.

The proof is trivial.

**REFERENCE**