ON ASYMPTOTIC BEHAVIORS OF ANALYTIC MAPPINGS
ON THE MARTIN BOUNDARY

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ABSTRACT. Some generalizations of the analogue of Collingwood and Cartwright in the large of Iversen’s theorem are given.

Let $f$ be a nonconstant analytic mapping of a hyperbolic Riemann surface $R$ into a Riemann surface $R'$. Let $R^*$ and $R'^*$ denote the Martin compactification and any compactification of $R$ and $R'$, respectively. $\Delta$ and $\Delta'$ denote the Martin ideal boundary of $R$ and the ideal boundary of $R'$, respectively. $\overline{\Delta}$, $A^c$ and $\text{int} A$ mean the closure, the complement and the interior of a set $A$ ($\subset R^*$ or $R'^*$) with respect to $R^*$ or $R'^*$, respectively. Let $\partial A$ denote the relative boundary of $A$ ($\subset R$ or $R'$) with respect to $R$ or $R'$ and $f_G$ the restriction of $f$ to $G$ ($\subset R$).

Let $\{G_n^{(e)}\}$ be a determinant sequence of Kerékjártó-Stoilow’s ideal boundary point $e$, and set $\Delta^e_n = \bigcap_n G_n^{(e)}$ and $\Delta^{G_n^{(e)}} = G_n^{(e)} \cap \Delta$. The cluster set of $f$ for $\Delta^e_n$ is defined by $C(f, \Delta^e_n) = \bigcap_n \overline{f(G_n^{(e)})}$, and the range of $f$ for $\Delta^e_n$ by $R(f, \Delta^e_n) = \bigcap_n f(G_n^{(e)})$.

In this paper we assume that the harmonic measure of $\Delta^e_n$ is positive.

For $b \in \Delta_1$, let $F_b$ be a filter basis on $R$ with respect to the fine topology, and set $\hat{f}(b) = \bigcap_{U \in F_b} \overline{f(U)}$. Here $\Delta_1$ denotes the set of minimal points in $\Delta$. If $\hat{f}(b)$ consists of a single point, then $f(b)$ is denoted by $\hat{f}(b)$.

We say that a curve $p = (t) (0 < t < 1)$ on $R$ converges to $e$, when for every $n$, there exists $\ell(n)$ such that $\lambda(t) \subset G_n^{(e)}$ for all $t \geq \ell(n)$. $\lambda(t)$ denotes the end of this path: $p = (t) (0 < t < 1)$. Let $\Gamma(f, \Delta_G^{(e)})$ denote the set of asymptotic points along all the paths such that the end of each path is on $\Delta_G^{(e)}$, and set $\chi(f, \Delta^e_n) = \bigcap_n \Gamma(f, \Delta_G^{(e)})$ and $\chi^*(f, \Delta^e_n) = \bigcap_n \Gamma(f, \Delta_G^{(e)})$. If for any neighborhood $V$ of $\alpha \in R^*$, $V \cap \Gamma(f, \Delta_G^{(e)})$ is a nonpolar set, we say $\alpha \in \Gamma(f, \Delta^e_n)$ and set $\chi^*(f, \Delta^e_n) = \bigcap_n \Gamma(f, \Delta_G^{(e)})$.

**Lemma 1.** If $\alpha \in \chi^*(f, \Delta^e_n) \cap C(f, \Delta^e_n) \cap R'$, then $\alpha \in \text{int} R'(f, \Delta^e_n)$.

**Proof.** Since $\alpha \in \chi^*(f, \Delta^e_n) \cap R'$, there exist a parametric disk $V$ about $\alpha$ and $G_n^{(e)}$ such that $V \cap \Gamma(f, \Delta_G^{(e)})$ is a polar set. Let $w = \psi(q)$.
Let \( q \in V \) be a local parameter of \( V \), and we set \( \psi(V) = \{w; |w| < 1\} \) \( \psi(\alpha) = 0 \), \( W_r = \{w; |w| < r, 0 < r < 1\} \), \( C_r = \partial W_r \), and \( \psi \circ f_{G_N}(e) = g \).

Since \( W_1 \cap \Gamma(g, \Delta_{G_N}^{(e)}) \) is a polar set, its linear measure is zero. Hence \( \Gamma(g, \Delta_{G_N}^{(e)}) \cap C_r = \emptyset \) for almost all \( r \) in \( 0 < r < 1 \). Let \( C_r \) have this property and fix \( r \).

Since \( \alpha \in \mathcal{C}(f, \Delta_e) \), we see that \( g^{-1}(W_r) \cap \mathcal{G}^{(e)}_n \neq \emptyset \) for all \( n \geq N \).

If there exists \( G \in \mathcal{G}^{(e)}_n \) such that \( G \subset \mathcal{G}_N^{(e)} \) and \( g(G) \subset W_r \), then for each point \( b \) of a set \( H_e \subset \Delta_1 \cap \Delta_e \) whose harmonic measure is positive, \( \hat{g}_G(b) \in \overline{W}_r \). Indeed the harmonic measure of \( \Delta_e \) is positive and \( g_G \) is a Fatou mapping of \( G \) into \( W_r \).

Hence there exists an asymptotic path \( \gamma \) from a point of \( G \) to each point \( b \) of \( H_e \) such that \( \lim_{p \to \gamma \cap \Delta_e} \phi_G(p) = \hat{g}_G(b) \). On the other hand, since \( \overline{W}_r \cap \Gamma(g, \Delta_C) \subset \hat{g}_G(b) \), for \( b \in H_e \) is a polar set, the harmonic measure of \( H_e \) is zero. This is a contradiction. Thus for all \( n \geq N \), we conclude that \( \mathcal{G}^{(e)}_n \cap \partial g^{-1}(W_r) \neq \emptyset \).

If \( \partial g^{-1}(W_r) \) contains closed Jordan curves accumulating to \( e \), then we see easily that \( w \in R(g, \Delta_e) \) for any point \( w \) on \( C_r \).

If for all \( n \geq N \), \( \mathcal{G}^{(e)}_n \) has at least one noncompact \( \gamma_\infty \) of \( \partial g^{-1}(W_r) \), let \( z = \phi(p) \) be a local parameter about \( p \in \mathcal{G}_N^{(e)} \), and set \( h = g \circ \phi^{-1} \). A function element \( Q(w) \) of \( z = h^{-1}(w) \) can be continued analytically along \( C_r \) infinitely often. Indeed if not, when \( w \) tends to a point \( w_1 \) \( \in \mathcal{C}_r \) along \( C_r \), \( \gamma_n \) is a path whose end is on \( \Delta_{G_n^{(e)}} \), and so \( w_1 \in \Gamma(g, \Delta_{G_n^{(e)}}) \). This is a contradiction.

Therefore any point \( w \) on \( C_r \) corresponds to an infinite number of points on \( \gamma_\infty \) for any \( n \), and hence \( w \in R(g, \Delta_e) \).

Therefore since for any point \( p \) of \( W_1 \), any neighborhood \( (cW_1) \) of \( p \) contains points of \( R(g, \Delta_e) \), we get \( W_1 \subset R(g, \Delta_e) \) and \( \alpha \in \operatorname{int} R(f, \Delta_e) \), as claimed.

**Corollary 1.** If \( \mathcal{C}(f, \Delta_e) \) is nowhere dense, then \( \mathcal{C}(f, \Delta_e) \cap R' \subset \mathcal{X}(f, \Delta_e) \).

**Proof.** If \( \alpha \in \operatorname{int} R(f, \Delta_e) \), for a neighborhood \( V \) of \( \alpha \), any neighborhood \( (cV) \) of any point \( \beta \in V \) contains at least one point of \( R(f, \Delta_e) \) and \( \mathcal{C}(f, \Delta_e) \) is not nowhere dense. Thus we have \( \mathcal{C}(f, \Delta_e) \cap R' \subset \mathcal{X}(f, \Delta_e) \).

**Lemma 2.** If \( \alpha \in \mathcal{X}(f, \Delta_e) \cap \mathcal{X}(f, \Delta_e) \cap \mathcal{C}(f, \Delta_e) \cap R' \), then \( \alpha \in R(f, \Delta_e) \).

**Proof.** Suppose that \( \alpha \notin R(f, \Delta_e) \). Since \( \alpha \in \mathcal{X}(f, \Delta_e) \cap \mathcal{X}(f, \Delta_e) \cap \mathcal{C}(f, \Delta_e) \cap R' \), there exist a parametric disk \( U \) about \( \alpha \) and \( G_N^{(e)} \) such that \( U \cap \Gamma(f, \Delta_{G_N^{(e)}}) \) is a polar set and \( \alpha \notin \Gamma(f, \Delta_{G_N^{(e)}}) \).

All the \( \alpha \)-points of \( f_{G_N}(e) \) are contained in a finite set of parametric
disks $\{U_k\}$ ($k = 1, 2, \ldots, L$) such that $U_i \cap U_j = \emptyset$ ($i \neq j$). Let $V$ be a parametric disk about $\alpha$ satisfying $V \subset \left( \bigcap_{k=1}^{L} \Gamma_{G_N}^e(U_k) \right) \cup U$. We fix $r$ such that $\Gamma(g, \Delta_{G_N}^e) \cap C_r = \emptyset$. There exists a diameter $d_r$ of $W_r$ such that $\Gamma(f, \Delta_{G_N}^e) \cap d_r = \emptyset$. There exists a diameter $d_r$ of $W_r$ such that $\Gamma(f, \Delta_{G_N}^e) \cap d_r = \emptyset$.

Since $b \in R(f, \Delta_r^e)$ for $b \in C_r$, there exists a connected component $D$ of $g^{-1}(W_r)$ which is not relatively compact. Choose a point $p$ on $\partial D$ which is mapped by $g$ to an endpoint of $d_r$. The function element $Q(w)$ corresponding to $p$ can be continued analytically along $d_r$ through the point $0$ to the antipodal point and $d_r$ is mapped on a cross-cut of $D$. But on the other hand, since $D$ does not contain the zeros of $g$, we have a contradiction, and we conclude that $a \in R(f, \Delta_r^e)$.

**Theorem 1.** If $R^*$ is a metrizable and resolutive compactification of $R'$ and, for at least one $n$, $\Gamma(f, \Delta_{G_N}^e)$ is a polar set, then $R(f, \Delta_e^c) \cap R' \subset \chi(f, \Delta_e)$.

**Proof.** From Lemma 2, we have $R(f, \Delta_e^c) \cap R' \subset C(f, \Delta_e) \cup \chi(f, \Delta_e)$.

If $C(f, \Delta_e^c) \neq \emptyset$, there exist a parametric disk $V$ and $G \in \{G_n^e\}$ ($G \subset G_n^e$) such that $f(G) \cap \overline{V} = \emptyset$. Since the mapping $f_G$ of $G$ into $R' - V$ is a Fatou mapping, it contradicts that the harmonic measure of $H_e$ is positive, as we see from the proof of Lemma 1.

Thus from $\Gamma(f, \Delta_{G_n}^e) = \emptyset$, we have $R(f, \Delta_e^c) \cap R' \subset \chi(f, \Delta_e)$.

**Lemma 3.** If $\alpha \in \chi^*(f, \Delta_e^c) \cap C(f, \Delta_e) \cap R'$, then $\alpha \in \text{int} R(f, \Delta_e)$.

**Proof.** In Lemma 1, take "all $r$ in $0 < r < 1$" instead of "almost all $r$ in $0 < r < 1$" and consider "$W_1 \cap \Gamma(g, \Delta_{G_N}^e) = \emptyset$" instead of "$W_1 \cap \overline{\Gamma(g, \Delta_{G_N}^e)}$ is a polar set". Then we have $w \in R(g, \Delta_e)$ for all $w: 0 < |w| < 1$ as in the proof of Lemma 1.

If $w_0 \in W_{r/2}^e$ ($w_0 \neq 0$), we have $w_0 \in C(g, \Delta_e)$ and $W_{r/2}^e \cap \Gamma(g, \Delta_{G_N}^e)$ = $\emptyset$ ($W_{r/2}^e = \{w; |w - w_0| < r/2\}$), and hence $0 \in R(g, \Delta_e)$.

Thus we have $W_1 \subset R(g, \Delta_e)$ and $\alpha \in \text{int} R(f, \Delta_e)$.

**Theorem 2.** $R(f, \Delta_e^c) \cap C(f, \Delta_e) \cap R' \subset \chi^*(f, \Delta_e)$.

**Proof.** From Lemma 3, we have

$\chi^*(f, \Delta_e^c) \subset C(f, \Delta_e^c) \cup R^e \cup (\text{int} R(f, \Delta_e))$;

that is,
Lemma 4. \( \text{int} C(f, \Delta_e) \subseteq \overline{R(f, \Delta_e)} \).

Proof. If \( \alpha \in \text{int} C(f, \Delta_e) \), for any neighborhood \( U \) of \( \alpha \), there exists a parametric disk \( V_0 \) about \( \alpha_0 \) satisfying \( V_0 \subseteq U \cap C(f, \Delta_e) \). Since \( \alpha_0 \in C(f, \Delta_e) \), there exists \( p_1 \in G_1^{(e)} \) such that \( \alpha_1 = f(p_1) \in V_0 \). We can take a parametric disk \( V_1 \) about \( \alpha_1 \) satisfying \( V_1 \subseteq V_0 \cap f(G_1^{(e)}) \). Repeating the same method, we have a sequence of parametric disks \( \{V_n\} (n = 1, 2, 3, \ldots) \) such that \( V_{n+1} \subseteq V_n \) and \( V_n \subseteq f(G_n^{(e)}) \). \( \beta \in \bigcap_n V_n \) is assumed by \( f \) in any \( G_n^{(e)} \), and hence \( \alpha \in \overline{R(f, \Delta_e)} \).

Corollary 2. \( R(f, \Delta_e)^c \cap R' \subseteq \chi^*(f, \Delta_e) \) if and only if \( R(f, \Delta_e) = R'^* \).

Proof. If \( C(f, \Delta_e) \neq R'^* \), there exists \( \alpha_0 \) such that \( \alpha_0 \in C(f, \Delta_e)^c \cap R' \subseteq R(f, \Delta_e)^c \cap R' \). If \( \alpha \in \chi^*(f, \Delta_e) \), then we have \( \alpha \in \overline{R(f, \Delta_e)} \) for any \( n \) and \( 0 \in \overline{\Gamma(g, \Delta_n^{(e)})} \) for a parametric disk \( V \) about \( \alpha \). Since there exists \( \omega_n \in W_{1/n} \cap \overline{\Gamma(g, \Delta_n^{(e)})} \), there exists \( p_n \in G_n^{(e)} \) such that \( g(p_n) \in W_{1/n} \). Since \( p_n \) converges to \( e \) and \( g(p_n) \) converges to \( 0 \), we see that \( 0 \in C(g, \Delta_e) \) and \( \alpha \in C(f, \Delta_e) \). Hence we have \( \chi^*(f, \Delta_e) \subseteq C(f, \Delta_e) \) and \( \alpha_0 \notin \chi^*(f, \Delta_e) \). Thus if \( R(f, \Delta_e)^c \cap R' \subseteq \chi^*(f, \Delta_e) \), from Lemma 4, \( \overline{R(f, \Delta_e)} = R'^* \).

Conversely if \( \overline{R(f, \Delta_e)} = R'^* \), then we have, from Theorem 2,

\[
R(f, \Delta_e)^c \cap R' = R(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R' \subseteq \chi^*(f, \Delta_e).
\]

Corollary 3. If the characteristic function of \( f \) (cf. [3]) is unbounded, then \( R(f, \Delta_e)^c \cap R' \subseteq \chi^*(f, \Delta_e) \).

Proof. If \( C(f, \Delta_e) \neq R'^* \), since \( f \) is a Lindelöf mapping, as in the proof of Theorem 1, the characteristic function of \( f \) is bounded, and a contradiction. Thus from Lemma 4 and Corollary 2 we get \( R(f, \Delta_e)^c \cap R' \subseteq \chi^*(f, \Delta_e) \).

REFERENCES