

BIRELATIONS AND SIDON SETS¹

S. W. DRURY

ABSTRACT. It is shown that a knowledge of the set of birelations with coefficients $-1, 0, 1$ supported on a subset of a discrete abelian group is enough to determine whether the subset has the Sidon property or not.

1. **Statement of the result.** Let X be a subset of a discrete abelian group G . We say that X is Sidon if every bounded complex-valued function on X is the restriction to X of a Fourier-Stieltjes transform on G . We refer the reader to [1] for the elementary theory of Sidon sets, unexplained notations and definitions.

Given complete knowledge of the group of relations on X it is possible in theory to determine whether X is Sidon or not. How to do this 'in practice' is an open question. The object of this note is, roughly speaking, to show that complete knowledge of the set of relations on X with coefficients $-1, 0, 1$ is in theory enough to decide whether or not X is Sidon. It should be noted that many of the classical combinatorial arithmetic conditions sufficient for X to be Sidon depend only on relations with small coefficients [1], [3].

In fact the concept of a relation is a bad one in this context since only certain relations count—the birelations. We view a birelation as a function $n: X \rightarrow \mathbf{Z}$ of finite support such that

$$(1) \quad \sum_{x \in X} n(x) \cdot x = O_G, \quad \sum_{x \in X} n(x) = O_{\mathbf{Z}}.$$

Indeed we view the group of all functions $n: X \rightarrow \mathbf{Z}$ of finite support (under pointwise operations) as the free abelian group generated by X . The canonical embedding $j: X \rightarrow \text{free } X$ is given by

$$(j(x))(y) = 1 \quad \text{if } x = y, \\ = 0 \quad \text{if } x \neq y.$$

We denote by Q_N the subset of free X given by $\{n; n \in \text{free } (X), |n(x)| \leq N \forall x \in X\}$.

Let X be an abstract set embedded in discrete abelian groups G_k ($k = 1, 2$) in two (different) ways. Let R_k be the corresponding groups of birelations in free X .

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Theorem. *If $R_1 \cap Q_1 = R_2 \cap Q_1$ then either X is Sidon in both embeddings or X is non-Sidon in both embeddings.*

2. **Reduction to finite sets X .** If X is Sidon then X is Sidon (α) for some α ($0 < \alpha \leq 1$). Here Sidon (α) is characterized by one of the following equivalent definitions.

$$(2) \quad \alpha \|\phi\|_{B(X)} \leq \|\phi\|_{C(X)} \quad \forall \phi \in C(X),$$

$$(3) \quad \|\lambda\|_{p,M} \geq \alpha \|\lambda\|_M \quad \forall \lambda \in M(X).$$

To prove the Theorem it suffices to show that if $R_1 \cap Q_1 = R_2 \cap Q_1$ and X is Sidon in G_1 , then X is Sidon in G_2 . Thus, by reason of the fact that X is Sidon (α) if and only if all its finite subsets are, we need only show the following in the case $N = 1$.

Proposition. $\forall \alpha$ ($0 < \alpha \leq 1$), $\forall N \geq 1 \exists \beta = \beta(\alpha, N)$ with $0 < \beta \leq 1$ such that if X is finite Sidon (α) in G_1 and $R_1 \cap Q_N = R_2 \cap Q_N$ then X is Sidon (β) in G_2 . (Actually one may show $\beta \geq \alpha(1 - 1/N)$ if $N \geq 2$.)

3. **A new embedding.** The compact dual group of free X , denoted Ω , is identified to the group of unimodular functions on X under pointwise multiplication and convergence, the duality being given by

$$\langle \omega, n \rangle = \prod_{x \in X} \omega(x)^{n(x)}.$$

The annihilator of the birelations subgroup R in Ω is denoted Γ and identified to

$$\Gamma = \{z \cdot \chi|_X; z \in \mathbb{C}, |z| = 1, \chi \in \hat{G}\}.$$

We may evaluate the elements of Γ at points of X and these evaluations are distinct. Thus X may be embedded as a subset of $\hat{\Gamma}$. This embedding has the same birelations as the original embedding. It has no relations which are not birelations. In particular $O_{\hat{\Gamma}}$ does not lie in X .

The following lemma is an immediate consequence of criterion (3).

Lemma 1. *If X is Sidon (β) in $\hat{\Gamma}$ then X is Sidon (β) in G .*

In the opposite direction we have the following more technical result.

Lemma 2. *Let X be a finite Sidon (α) subset of G and let $\epsilon > 0$. Then there exist positive measures μ_ω on Γ such that*

- (a) $\|\mu_\omega\|_M = (1 + \epsilon)\alpha^{-1} \forall \omega \in \Omega$,
- (b) $\hat{\mu}_\omega(x) = \omega(x) \forall x \in X \forall \omega \in \Omega$,
- (c) $\omega \rightarrow \mu_\omega$ is norm continuous on Ω .

Here (b) is to be interpreted relative to the new embedding. As there is little new in this lemma we merely sketch the proof.

Firstly let $\text{co}(\Omega)$ denote the convex hull of Ω that is the set of maps of X into the closed unit disc. For each $\phi \in \text{co}(\Omega)$ we show $\exists \mu_\phi \in M^+(\Gamma)$ such that

- (a₁) $\|\mu_\phi\|_M \leq (1 + \epsilon)\alpha^{-1}$,
- (b₁) $\hat{\mu}_\phi(x) = \phi(x) \forall x \in X$.

Let $C = \{\hat{\mu}|_X; \mu \in M^+(\Gamma) \text{ satisfies (a}_1)\}$, a closed convex subset of $C(X)$. It remains to show that $\phi \in C$. If not, by Hahn-Banach $\exists \lambda \in M(X)$ such that

$$(4) \quad \Re(\lambda, \phi) \geq 1,$$

$$(5) \quad \Re(\lambda, u) \leq 1 \quad \forall u \in C.$$

But (4) implies that $\|\lambda\|_M \geq 1$ and (5) that $(1 + \epsilon)\|\lambda\|_{PM} \leq \alpha$ which together contradict (3).

Secondly we show that for each $\phi \in \text{co}(\Omega)$, $\exists \mu_\phi \in M^+(\Gamma)$ such that (a₁),

$$(c_2) \quad \begin{aligned} &|\hat{\mu}_\phi(x) - \phi(x)| \leq \epsilon \quad \forall x \in X, \\ &\phi \rightarrow \mu_\phi \text{ is norm continuous on } \text{co}(\Omega) \end{aligned}$$

hold. Indeed this follows from a routine 'partition of unity' argument on $\text{co}(\Omega)$.

Thirdly we show that for each $\phi \in \text{co}(\Omega) \exists \mu_\phi \in M^+(\Gamma)$ such that (a₁), (b₁) and (c₂) hold. This follows from a routine iteration argument (which involves a change in ϵ).

Fourthly we show that for each $\phi \in \text{co}(\Omega) \exists \mu_\phi \in M^+(\Gamma)$ such that $\|\mu_\phi\|_M = (1 + \epsilon)\alpha^{-1}$, (b₁) and (c₂) hold. This follows by adding in the measure $((1 + \epsilon)\alpha^{-1} - \|\mu_\phi\|_M)\eta_\Gamma$, where η_Γ denotes the Haar measure on Γ . Since $O_{\hat{\Gamma}}$ does not lie in X as embedded in $\hat{\Gamma}$, this does not affect the value of $\hat{\mu}_\phi$ on X .

Finally restricting the choice of ϕ to Ω gives the required result.

4. Proof of Proposition 1 in the case $N = 2$. We deal with this case first as it is the easiest to understand.

Let $\epsilon > 0$ be arbitrary. For $\rho \in \Omega$ let μ_ρ denote the measure of Lemma 2 applied to the embedding $X \subseteq G_1$. We define for $\omega \in \Omega$ and $\chi \in \Gamma_2$ the integrated Riesz product

$$R(\omega, \chi) = \iint \prod_{x \in X} (1 + \Re(\omega\rho^{-1}\chi\phi^{-1})(x)) d\mu_\rho(\phi) d\eta_\Omega(\rho)$$

where η_Ω is Haar measure on Ω and ϕ denotes a generic element of Γ_1 . Clearly R is a nonnegative continuous function on $\Omega \times \Gamma_2$. Thus $\|R(\omega, \cdot)\|_1 = \hat{R}(\omega, O_{\hat{\Gamma}_2})$ and a calculation shows that this equals

$$(6) \quad \sum \iint \left(\frac{1}{2}\right)^{\Sigma |n(x)|} \langle \omega \rho^{-1} \phi^{-1}, n \rangle d\mu_\rho(\phi) d\eta_\Omega(\rho)$$

where the inner sum is taken over $x \in X$ and the outer one over $n \in Q_1 \cap R_2$. Since by hypothesis $Q_1 \cap R_2 = Q_1 \cap R_1 \subseteq R_1$, we have $\langle \phi^{-1}, n \rangle = 1$ in each term. The integrand with respect to μ_ρ is then constant and by (a) we have

$$\|R(\omega, \cdot)\|_1 = (1 + \epsilon)\alpha^{-1} \sum \int \left(\frac{1}{2}\right)^{\Sigma |n(x)|} \langle \omega \rho^{-1}, n \rangle d\eta_\Omega(\rho).$$

On integration by η_Ω the only term to give a contribution is the one corresponding to $n = 0$. Hence

$$\|R(\omega, \cdot)\|_1 = (1 + \epsilon)\alpha^{-1} \quad \forall \omega \in \Omega.$$

A similar calculation shows that for $y \in X$, $\hat{R}(\omega, y)$ equals (6) where the outer sum is now taken over $\{n; n \in Q_1, n - j(y) \in R_2\}$. For such n , $n - j(y) \in Q_2 \cap R_2 = Q_2 \cap R_1 \subseteq R_1$ so that in each term $\langle \phi^{-1}, n \rangle = \phi^{-1}(y)$. Using (b) we obtain

$$\hat{R}(\omega, y) = \sum \int \left(\frac{1}{2}\right)^{\Sigma |n(x)|} \rho(y) \langle \omega \rho^{-1}, n \rangle d\eta_\Omega(\rho) = \frac{1}{2} \omega(y).$$

It follows that X is Sidon ($\frac{1}{2}\alpha$) in $\hat{\Gamma}_2$ and hence also in G_2 by Lemma 1.

The particular estimates $\beta = (1 - 1/N)\alpha$ for general $N \geq 2$ are obtained by a similar argument applied to the corresponding Fejer kernel Riesz product

$$R(\omega, \chi) = \iint \prod_{x \in X} K_N((\omega \rho^{-1} \chi \phi^{-1})(x)) d\mu_\rho(\phi) d\eta_\Omega(\rho)$$

where $K_N(z) = \sum_{|n| \leq N} (1 - |n|/N)z^n$ denotes the Fejer kernel.

5. Proof of Proposition 1 in case $N = 1$. In this section we extend the proof of §4 by adapting two well-known ideas. The following variant of Lemma 2 is based on Drury [2, §2].

Lemma 3. *Let X be a finite Sidon (α) subset of G and let $\epsilon > 0$. Then there exist positive measures ν_ω on Γ such that*

- (a') $\|\nu_\omega\|_M = (1 + \epsilon)\alpha^{-2} \forall \omega \in \Omega$,
- (b') $\hat{\nu}_\omega(x) = \omega(x) \forall x \in X \forall \omega \in \Omega$,
- (c') $\omega \rightarrow \nu_\omega$ is norm continuous on Ω ,
- (d') $\int \langle \omega, n \rangle \hat{\nu}_\omega(u) d\eta_\Omega(\omega) \geq 0 \forall u \in \hat{\Gamma} \forall n \in \text{free } X$.

Proof. Let μ_ω be the measure of Lemma 2 and set $\nu_\omega = \int \mu_{\omega\lambda^{-1}} * \tilde{\mu}_\lambda d\eta_\Omega(\lambda)$ where \sim denotes the usual involution on $M(\Gamma)$ and λ is a generic element of Ω . After suitable juggling with ϵ , the first three conditions are

easily verified. Condition (d') follows since

$$\int \langle \omega, n \rangle \hat{\nu}_\omega(u) d\eta_\Omega(\omega) = \left| \int \langle \omega, n \rangle \hat{\mu}_\omega(u) d\eta_\Omega(\omega) \right|^2 \geq 0.$$

This completes the proof of Lemma 3.

With $\epsilon > 0, 0 < \delta \leq 1$ and the ν_ρ of Lemma 3, we define

$$R_\delta(\omega, \chi) = \iint \prod_{x \in X} (1 + \delta \Re(\omega \rho^{-1} \chi \phi^{-1})(x)) d\nu_\rho(\phi) d\eta_\Omega(\rho).$$

As in §4, R_δ is a nonnegative continuous function on $\Omega \times \Gamma_2$ satisfying

$$(7) \quad \|R_\delta(\omega, \cdot)\|_1 = (1 + \epsilon)\alpha^{-2} \quad \forall \omega \in \Omega.$$

In particular this holds for $\omega = 1$ and $\delta = 1$. Thus

$$(8) \quad |\hat{R}_1(1, y)| \leq (1 + \epsilon)\alpha^{-2} \quad \forall y \in X \subseteq \hat{\Gamma}_2.$$

The use of (8) in the control of $\hat{R}_\delta(\omega, y)$ stems from an idea of Rider [3, Lemma 1.7]. We have

$$(9) \quad \hat{R}_\delta(\omega, y) = \sum \left(\frac{\delta}{2}\right)^{\Sigma|n(x)|} a(n) \langle \omega, n \rangle$$

where the inner summation is over $x \in X$, the outer sum is over

$$S = \{n; n \in Q_1, n - j(y) \in R_2\}$$

and

$$a(n) = \iint \langle \rho^{-1} \phi^{-1}, n \rangle d\eta_\rho(\phi) d\eta_\Omega(\rho).$$

As in §4 we have $a(j(y)) = 1$. Thus by (9),

$$(10) \quad |\hat{R}_\delta(\omega, y) - \frac{1}{2}\delta\omega(y)| \leq \sum \left(\frac{\delta}{2}\right)^{\Sigma|n(x)|} |a(n)|$$

where the outer sum is now over $S \setminus \{j(y)\}$. By (1), if $n \in S$, then $\Sigma|n(x)|$ is odd. Since the points of X are distinct (in G_2) it follows that $\Sigma|n(x)| \geq 3 \forall n \in S \setminus \{j(y)\}$. Thus by (10), and since $\delta \leq 1$, we have

$$(11) \quad |\hat{R}_\delta(\omega, y) - \frac{1}{2}\delta\omega(y)| \leq \delta^3 \sum \left(\frac{1}{2}\right)^{\Sigma|n(x)|} |a(n)|,$$

the outer sum still being taken over $S \setminus \{j(y)\}$. By (d'), $a(n)$ is always non-negative. Together with (8) and (9) this shows that

$$\sum_S \left(\frac{1}{2}\right)^{\Sigma|n(x)|} |a(n)| \leq (1 + \epsilon)\alpha^{-2}.$$

Using this estimate in (11) we have

$$|\hat{R}_\delta(\omega, y) - \frac{1}{2}\delta\omega(y)| \leq \delta^3((1 + \epsilon)\alpha^{-2} - \frac{1}{2}).$$

This together with (7) and the standard iteration argument shows that X is

Sidon (β) in $\hat{\Gamma}_2$ and hence also in G_2 . For an arbitrarily small value of ϵ and $\delta = 3^{-1/2}\alpha(2 - \alpha^2)^{-1/2}$ we may take β to be $3^{-3/2}\alpha^3(2 - \alpha^2)^{-1/2}$.

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DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL 101, CANADA