CHARACTERIZING $\text{hol}(\Omega)$
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ABSTRACT. We find necessary and sufficient conditions for an algebra of complex valued functions defined on a $\sigma$-compact $T_2$ space to be algebraically and topologically equivalent to the algebra of analytic functions on a finitely connected domain in $\mathbb{C}$.

Let $\Omega$ be a finitely connected open set in the complex plane $\mathbb{C}$. Denote by $\text{hol}(\Omega)$ the set of all functions which are analytic on $\Omega$. Note that $\text{hol}(\Omega)$ is an algebra with respect to the operations of pointwise addition and multiplication of functions and scalar multiplication by complex numbers. Note also that $\text{hol}(\Omega)$ is a Fréchet space with respect to the topology of uniform convergence on compact subsets of $\Omega$.

Let $X$ be a $\sigma$-compact $T_2$ space. That is, $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i$ is a compact subset of $X$. Let $A$ be an algebra of complex valued continuous functions on $X$ which is complete with respect to the topology of uniform convergence on the sets $X_i$ and contains the constant functions. We give conditions on $A$ which are necessary and sufficient for $A$ to be topologically and algebraically isomorphic to $\text{hol}(\Omega)$ for some finitely connected open set $\Omega$ in $\mathbb{C}$.

In order to find conditions on $A$ which are natural, we will first consider $\text{hol}(\Omega)$ and see what kind of conditions this algebra admits. If $C_1, \ldots, C_n$ are the bounded components of $\mathbb{C} - \Omega$ and $z_i$ is in $C_i$, then polynomials in the functions $f, f_1, \ldots, f_n, f(z) = z$ and $f_i(z) = (z - z_i)^{-1}$ are dense in $\text{hol}(\Omega)$.

A derivation on an algebra $B$ is a linear function $d: B \rightarrow B$ which satisfies the multiplicative condition $d(ab) = d(a)b + ad(b)$ for every $a$ and $b$ in $B$. There is a natural derivation on $\text{hol}(\Omega)$ given by $d(g) = g'$ ($g'$ denotes the ordinary derivative of $g$).

We will show that the conditions outlined in the above two paragraphs are necessary and sufficient for the algebra $A$ to be topologically and algebraically isomorphic to $\text{hol}(\Omega)$. This result is given in the following

**Theorem.** The following two conditions are necessary and sufficient for $A$ to be topologically and algebraically isomorphic to $\text{hol}(\Omega)$ for some finitely connected open set $\Omega$ in $\mathbb{C}$.

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1. There is a function $f$ in $A$ and complex numbers $z_1, \ldots, z_n$ in the resolvent set of $f$ such that polynomials in $f$, $(f - z_1)^{-1}, \ldots, (f - z_n)^{-1}$, are dense in $A$.

2. There is a derivation $d: A \to A$ such that $d(f)$ is never zero on $X$.

**Proof.** We noted in the paragraphs preceding the Theorem that conditions 1 and 2 are necessary. We will prove they are also sufficient.

Without loss of generality we can assume $X = \bigcup X_i$ where $X_i$ is compact and $X_i \subset X_{i+1}$. We let $M$ denote the collection of all continuous homomorphisms from $A$ onto $C$. Note that $M = \bigcup M_i$ where $M_i$ is the set of all homomorphisms $m$ in $M$ which satisfy the continuity condition $|m(g)| \leq \sup_{x \in X_i} |g(x)|$ for every $g$ in $A$. For each $g$ in $A$ define a function $g: M \to C$ by $g(m) = \hat{g}(m) = m(g)$ (g is the Gelfand transform of $g$). Since for each $x$ in $X$ the map $g \to g(x)$ is a homomorphism of $A$ onto $C$, we have that $A$ is topologically and algebraically isomorphic to the algebra $\hat{A}$ consisting of all functions of the form $\hat{g}$, $g$ in $A$ and having the topology of uniform convergence on the sets $\{M_i\}$.

Now consider the set $\hat{f}(M)$. It follows from the continuity of the elements of $M$ and the fact that $\hat{A}$ is generated by $f, f_1, \ldots, f_n$ that $\hat{f}$ is one-to-one. We define a set $\Omega$ in $C$ by $\Omega = \hat{f}(M)$. For each $g$ in $A$ define a function $\hat{g}$ on $\Omega$ by $\hat{g}(\hat{z}) = \hat{g}(\hat{f}^{-1}(z))$. Let $\hat{A}$ denote the algebra of all functions $\hat{g}$, $g$ in $A$. The map $g \to \hat{g}$ is a topological and algebraic isomorphism of $A$ onto $\hat{A}$ when $A$ is given the topology of uniform convergence on the sets $\{M_i\}$.

The algebra $\hat{A}$ is generated by the functions $f, f_1, \ldots, f_n$. For the function $f$ we have $\hat{f}(z) = \hat{f}(\hat{f}^{-1}(z)) = z$. Hence $\hat{A}$ is the completion of the polynomials in $z, (z - z_1)^{-1}, \ldots, (z - z_n)^{-1}$ on $\Omega$ with respect to the topology of uniform convergence on the sets $\{\hat{f}(M_i)\}$. If we can show that $\Omega = \bigcup \text{int } \hat{f}(M_i)$, then we will have that $\Omega$ is open and that the topology of uniform convergence on the sets $\{\hat{f}(M_i)\}$ is the same as the topology of uniform convergence on compact subsets of $\Omega$. It will then follow that $\hat{A} \subset \text{hol} (\Omega)$. We can show that each bounded component of $C - \Omega$ contains one of the points $z_i$, $i = 1, \ldots, n$, then we can conclude that $\hat{A} \subset \text{hol} (\Omega)$.

We will first show that $\Omega = \bigcup \text{int } \hat{f}(M_i)$. Assume the contrary. That is, there is a $z_0$ in $\Omega - \bigcup \text{int } \hat{f}(M_i)$. Using the identification we have made between $A$ and $\hat{A}$, we can regard $d$ as a derivation on $\hat{A}$. We proved in a previous paper [2] that such a derivation is necessarily continuous. Let $\|\hat{g}\|_p = \max |\hat{g}(\hat{f}(M_n))|$. Since $d$ is linear and continuous, there is a natural number $p$ and a constant $K$ such that $|d(\hat{g})(z_0)| \leq K\|\hat{g}\|_p$ for every $\hat{g}$ in $\hat{A}$. The set $\hat{f}(M_p)$ is the spectrum of $f$ in the algebra which is the completion of $A$ with respect to the sup-norm on $X_p$. Hence $\hat{f}(M_p)$ is a compact subset.
of $C$ with at most $n + 1$ components in its complement and each of the bounded components contains one of the points $z_i$, $i = 1, \ldots, n$. Let $b$ denote the point derivation defined on $\hat{A}$ by $b(g) = d(\hat{g})(z_0)$. Since $|d(\hat{g})(z_0)| < K||\hat{g}||_p$ for all $\hat{g}$ in $\hat{A}$, we can extend $b$ to a continuous derivation on $\hat{A}_p$, the completion of $\hat{A}$, with respect to the seminorm $||\cdot||_p$.

Since $z_0$ is a boundary point of $\hat{f}(M_p)$, there is a function $g_0$ in $C(\hat{f}(M_p))$ such that $g_0$ is analytic on $\text{int } \hat{f}(M_p)$ and $|g_0(z_0)| > |g_0(z)|$ for all $z \neq z_0$ in $\hat{f}(M_p)$. It follows from the properties of rational approximation (see [3]) that $g_0$ can be uniformly approximated on $\hat{f}(M_p)$ by polynomials in $f, f_1, \ldots, f_n$. Therefore $g_0$ is in $\hat{A}_p$. Since $g_0$ peaks at the point $z_0$, we have that $b$ must be zero on all of $\hat{A}_p$ (see [1, Corollary 1.6.7]). Since $\hat{f}(M_p)$ is the spectrum of $f$ with respect to the algebra $A_p$ and $z_0$ is in the boundary of $\hat{f}(M_p)$, there is a $x_0$ in $X$ such that $z_0 = \hat{f}(x_0)$. We have shown that $0 = b(f) = d(f)(x_0)$, contradicting our assumption that $d(f)$ is never zero on $X$. Therefore we must have $\Omega = \bigcup \text{int } \hat{f}(M_j)$ and $A \subset \text{hol}(\Omega)$.

All that remains is to show that each bounded component of $C - \Omega$ contains one of the points $z_i$, $i = 1, 2, \ldots, n$. Suppose there is a bounded component $C$ which does not contain any of the $z_i$. The boundary of $C$ is a compact set. Since $\Omega = \bigcup \text{int } \hat{f}(M_j)$, we have that $C \subset \hat{f}(M_j)$ for some $j$. The functions in $\hat{A}$ have unique analytic extensions to $\Omega \cup C$ and $\max|g(\hat{f}(M_j) \cup C)| = \max|g(\hat{f}(M_i))|$ for every $g$ in $\hat{A}$. Let $y$ be a point in $C$. The homomorphism $m: A \rightarrow C$, defined by $m(g) = g(y)$ for $g$ in $A$, is continuous. This contradicts the fact that $\Omega = \hat{f}(M)$. Hence, every component of $C - \Omega$ contains one of the $z_i$, and we have $A = \text{hol}(\Omega)$.

REFERENCES


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