A TOTALLY REAL SURFACE IN $CP^2$ THAT IS NOT TOTALLY GEODESIC

GERALD D. LUDDEN, MASAFUMI OKUMURA AND KENTARO YANO

ABSTRACT. An example of a totally real surface immersed in complex projective space is given. This surface is not totally geodesic. The relation of this example to previous theorems on totally real submanifolds is given.

1. Introduction. Let $CP^m$ denote $m$-dimensional complex projective space, normalized so that $CP^m$ is of constant holomorphic sectional curvature $4$. Let $J$ denote the almost complex structure of $CP^m$. A submanifold $M$ immersed in $CP^m$ is said to be totally real if $M_x \cap JM_x = \{0\}$ for all $x$ in $M$, where $M_x$ denotes the tangent space to $M$ at $x$. There have recently been several papers concerned with totally real submanifolds of complex manifolds (see [1], [2], [4], [8]). In particular, Chen and Ogiue [2] have proved

**Theorem 1.** Let $M$ be a compact $n$-dimensional manifold isometrically immersed in $CP^n$ as a minimal, totally real submanifold. If $\|\sigma\|^2 < (n+1)/(2 - 1/n)$, then $M$ is totally geodesic. Here $\sigma$ is the second fundamental form of the immersion.

The purpose of this paper is to show that there exist submanifolds of $CP^n$ for which the above inequality becomes equality and to determine such submanifolds. The existence of such submanifolds relies on the examples of minimal submanifolds of spheres given in Chern-do Carmo-Kobayashi [3].

2. An example of a totally real submanifold. Let $S^p(r)$ denote the Euclidean sphere of dimension $p$ and radius $r$. We denote $S^p(1)$ by just $S^p$. It is now well known that $S^{2m+1}$ is a principal circle bundle over $CP^m$ [7]. This is the Hopf fibration. If we consider $S^{2m+1}$ as a hypersurface of the Euclidean space $E^{2m+2}$ equipped with its natural Kaehler structure $\bar{J}$, then $\xi = \bar{J}N$, where $N$ is the outward unit normal to $S^{2m+1}$, is a unit tangent vector field on $S^{2m+1}$ whose integral curves are great circles. These great circles are the fibres of the Hopf fibration. Let $\pi: S^{2m+1} \rightarrow CP^m$ denote the submer-
as mentioned in [3], $S_1 \times S_2 \times S_3$ is naturally immersed in $S^5$ as a minimal submanifold. Of course, $S_1 \times S_2 \times S_3$ is a flat manifold and if $\bar{\sigma}$ is the second fundamental form of the immersion, it can be checked that (see [3]) $\|\bar{\sigma}\|^2 = 6$. If $N_a$ denotes the unit outward normal on $S_a$ ($a = 1, 2, 3$), then $N = (N_1, N_2, N_3)$ is the outward normal to $S^5$ at a point of $S_1 \times S_2 \times S_3$ and it is easy to see that $\xi = \bar{\sigma}N$ is tangent to $S_1 \times S_2 \times S_3$ at this point. Here $\xi$ is the natural almost complex structure on $E^6$. Since each of the factors $S_1, S_2, S_3$ can be assumed to lie in complex subspaces of $E^6$, just as in [5], there is a submanifold $T^2$ of $CP^2$ and a submersion $\pi : S_1 \times S_2 \times S_3 \rightarrow T^2$ such that the following diagram is commutative

$$
\begin{array}{ccc}
S_1 \times S_2 \times S_3 & \xrightarrow{i} & S^5 \\
\pi \downarrow & & \downarrow \bar{\pi} \\
T^2 & \xrightarrow{\iota} & CP^2
\end{array}
$$

It is also shown in [5] that $\iota$ is a minimal immersion.

Let $(\phi, \eta, \xi)$ denote the almost contact structure on $S^5$ induced from $\bar{\xi}$. Then

$$
\phi(\bar{\xi}(N_1, 0, 0)) = \bar{\iota}^2(N_1, 0, 0) - \eta(\bar{\iota}(N_1, 0, 0))N
$$

which is normal to $S_1 \times S_2 \times S_3$. Similarly $\phi(\bar{\xi}(0, N_2, 0))$ and $\phi(\bar{\xi}(0, 0, N_3))$ are normal to $S_1 \times S_2 \times S_3$. However, $\{\bar{\xi}(N_1, 0, 0), \bar{\xi}(0, N_1, 0), \bar{\xi}(0, 0, N_3)\}$ forms a basis of tangent vectors to $S_1 \times S_2 \times S_3$. We say $S_1 \times S_2 \times S_3$ is an anti-invariant (under $\phi$) submanifold of $S^5$. It can now be easily checked that $T^2$ is a totally real submanifold of $CP^2$ (see [6] for the local basis of all four manifolds that shows this). Since $T^2$ is totally real, formula (4.6) of [7] becomes $\|\bar{\sigma}\|^2 = \|\sigma\|^2 + 2 \cdot 2$, where $\sigma$ is the second fundamental form of the immersion $\iota$. Thus, $T^2$ is a compact, minimal, totally real surface immersed in $CP^2$ with $\|\sigma\|^2 = 2$. But this is precisely the case of equality in the theorem of Chen and Ogiue for $n = 2$.

Remark. Let $Z_a$ be the complex coordinates of $R^6$ ($a = 1, 2, 3$). Then $S^5 = \{(Z_1, Z_2, Z_3) : \|Z_1\|^2 = |Z_2|^2 + |Z_3|^2 = 1\}$ and $S_1 \times S_2 \times S_3 = \{(Z_1, Z_2, Z_3) : |Z_1|^2 = |Z_2|^2 = |Z_3|^2 = 1/3\}$. Basically $\bar{\iota}$ and $\iota$ are defined by the identification $(Z_1, Z_2, Z_3) \sim (Z_1/Z_3, Z_2/Z_3)$. Then $|Z_1/Z_3|^2 = |Z_2/Z_3|^2 = 1$ for $(Z_1, Z_2, Z_3)$ in $S_1 \times S_2 \times S_3$ so $T^2 = S^1 \times S^1$.

3. Main theorems. Let $M$ be an $n$-dimensional totally real, minimal submanifold of $CP^n$. We choose a local field of orthonormal frames $e_1, \ldots, e_{2n}$ in $CP^n$ such that, restricted to $M$, the vectors $e_1, \ldots, e_n$ are tangent to $M$ and $\omega^1, \ldots, \omega^{2n}$ is the field of dual frames. Then we have $d\omega^A = -\omega^B \wedge \omega^A$, $\omega^A_B + \omega^B_A = 0$ ($A, B = 1, \ldots, 2n$), from the structure equations of $CP^n$. Restrict-
ed to $M$, we have $\omega^{n+1} = \cdots = \omega^{2n} = 0$. Thus, $0 = \omega^a = -\sum \omega^a_i \wedge \omega^i$, and so by Cartan's lemma we have

$$
(1) \quad \omega^a_i = \sum h^a_{ij} \omega^i, \quad h^a_{ij} = h^a_{ji},
$$

where $i, j, \ldots = 1, \ldots, n$ and $\alpha, \beta, \ldots = n + 1, \ldots, 2n$. Also $\omega^a = -\sum \omega^a_i \wedge \omega^i$, $\omega^a_i+\omega^i = 0$. Since $M$ is minimal, $\sum h^a_{ij} = 0$ for all $a$. Also, if $\sigma$ is the second fundamental form of the immersion, then $\|\sigma\|^2 = \sum (h^a_{ij})^2$. Let $A_a$ denote the matrix formed from $h^a_{ij}$ and $\text{tr} A_a$ denote the trace of $A_a$. Then, by [2, Proposition 3.5], we have

$$
(2) \quad \frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla^\prime \sigma\|^2 + \sum \text{tr}(A^a A^b - A^b A^a) \|\sigma\|^2 - \sum (\text{tr} A^a A^b) \|\sigma\|^2 + (n + 1)\|\sigma\|^2,
$$

where $\Delta$ is the Laplacian operator and $\nabla^\prime$ is covariant differentiation in (tangent bundle) $\Theta$ (normal bundle).

We have the following lemma from [3].

**Lemma 1.** Let $A$ and $B$ be symmetric $(n \times n)$-matrices. Then $-\text{tr}(AB - BA)^2 \leq 2 \text{tr} A^2 \text{tr} B^2$, and the equality holds for nonzero matrices $A$ and $B$ if and only if $A$ and $B$ can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$
\tilde{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{pmatrix}.
$$

Moreover, if $A_1, A_2, A_3$ are $(n \times n)$-symmetric matrices, and if

$$
-\text{tr}(A^a A^b - A^b A^a)^2 = 2 \text{tr} A^a A^b, \quad 1 \leq a, b \leq 3,
$$

then at least one of the matrices $A_a$ must be zero.

Applying the inequality in the lemma to (2), we have

$$
\frac{1}{2} \Delta \|\sigma\|^2 \geq \|\nabla^\prime \sigma\|^2 - 2 \sum_{a \neq b} \text{tr} A^2 A^2 - \sum (\text{tr} A^a A^b) \|\sigma\|^2 + (n + 1)\|\sigma\|^2
$$

$$
(3) \quad = \|\nabla^\prime \sigma\|^2 - 2 \text{tr} A^2 A^2 + (n + 1)\|\sigma\|^4 + n(n - 1)(\sigma_1^2 - \sigma_2),
$$

where $n\sigma_1 = \sum \text{tr} A^2_a$, $n(n + 1)\sigma_2 = 2 \sum_{a < b} \text{tr} A^2_a \text{tr} A^2_b$ and $n^2(n - 1)(\sigma_1^2 - \sigma_2) = \sum_{a < b} (\text{tr} A^2_a - \text{tr} A^2_b)^2$. Using the inequality in Theorem 1 and Hopf's Lemma, it is now easy to see the proof of Theorem 1. Assume now that $\|\sigma\|^2 \equiv (n + 1)/(2 - 1/n)$. Then $\Delta \|\sigma\|^2 = 0$, and so by (3) we have

$$
(4) \quad -\text{tr}(A^a A^b - A^b A^a)^2 = 2 \text{tr} A^2_a \text{tr} A^2_b, \quad n + 1 \leq a \neq b \leq 2n,
$$

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(5) \[ \| \nabla' \sigma \|^2 = 0, \]
and
(6) \[ \text{tr} A^2_\alpha = \text{tr} A^2_\beta, \quad n + 1 \leq \alpha \neq \beta \leq 2n. \]

By Lemma 1, (4) implies that at most two of the $A_\alpha$'s are nonzero. However, by (6), if one $A_\alpha$ is zero then all the $A_\alpha$ are zero and $\sigma \equiv 0$, which is not the case. Thus $n \leq 2$. If $n = 1$ then $M$ is just a real curve in $\mathbb{CP}^1$ and totally real is no restriction. Thus, we will assume $n = 2$ and by Lemma 1 we have

\[ A_3 = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

and, by (5), we see that $\lambda$ and $\mu$ are constants. By (6) we have $\lambda^2 = \mu^2$, and since $\| \sigma \|^2 = 2$, we have $\lambda^2 + \mu^2 = 1$ so that $\lambda^2 = \frac{1}{2}$ and we may assume that $-\lambda = \mu = \frac{1}{\sqrt{2}}$.

Now $\omega^a_i = \sum h^a_{ij} \omega^j$ so that

\[ \omega^1 = \sum h^3_{1j} \omega^j = \lambda \omega^2, \quad \omega^2 = \sum h^3_{2j} \omega^j = \lambda \omega^1, \]
\[ \omega^3 = \sum h^4_{1j} \omega^j = \mu \omega^1, \quad \omega^4 = \sum h^4_{2j} \omega^j = -\mu \omega^2. \]

If we differentiate (1) and use (5), we obtain (see [3])

\[ 0 = -\sum h^a_{ii} \omega^i - \sum h^a_{ij} \omega^i + \sum h^b_{ij} \omega^b. \]

Setting $\alpha = 3$ and $i = j = 1$, this becomes

\[ 0 = -\sum h^3_{11} \omega^1 - \sum h^3_{11} \omega^1 + \sum h^3_{11} \omega^3 = -\lambda \omega^2 - \lambda \omega^1 + \mu \omega^3. \]

That is $\omega^3 = 2\lambda \mu \omega^1$. From formula (3.5) of [2] we see that $M$ is a flat manifold. Putting this all together we have

**Theorem 2.** Let $M$ be an $n$-dimensional $(n > 1)$, minimal, totally real submanifold of $\mathbb{CP}^n$ satisfying $\| \sigma \|^2 = (n + 1)/(2 - 1/n)$. Then $n = 2$ and, with respect to an adapted dual orthonormal frame field $\omega^1, \omega^2, \omega^3, \omega^4$, the connection form $(\omega^A_B)$ of $\mathbb{CP}^2$ restricted to $M$ is given by

\[ \begin{pmatrix} 0 & \omega^1_2 & -\lambda \omega^2 & -\mu \omega^1 \\
-\omega^1_2 & 0 & -\lambda \omega^1 & \mu \omega^2 \\
\lambda \omega^2 & \lambda \omega^1 & 0 & 2 \omega^1_2 \\
\mu \omega^1 & -\mu \omega^2 & -2 \omega^1_2 & 0 \end{pmatrix}, \quad -\lambda = \mu = \frac{1}{\sqrt{2}}. \]

Therefore such a submanifold is locally unique. Thus, we have

**Theorem 3.** If $M$ is a compact $n$-dimensional $(n > 1)$, minimal, totally real submanifold of $\mathbb{CP}^n$ satisfying $\| \sigma \|^2 = (n + 1)/(2 - 1/n)$, then $n = 2$ and $M = S^1 \times S^1$. 

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823

DEPARTMENT OF MATHEMATICS, SAITAMA UNIVERSITY, URAWA CITY, JAPAN

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO, JAPAN