PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

ON COMPACT EINSTEIN-KAEHLER MANIFOLDS

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ABSTRACT. A characterization of a complex space form among compact Einstein-Kaehler manifolds is given in terms of Chern classes.

1. **Introduction.** Let M be a Kaehler manifold and c_k the kth Chern class of M. A Kaehler manifold of constant holomorphic sectional curvature is call a *complex space form*. If M is an n-dimensional complex space form $(n \geq 2)$, then $c_2 = nc_1^2/2(n+1)$ (cf. [1]). In [1] we considered the converse.

The purpose of this paper is to study similar problems under somewhat weaker assumption.

Theorem. *Let M be an n-dimensional compact Kaehler manifold $(n \geq 2)$. If*

(i) $c_1^{n-2}c_2 = nc_1^n/2(n+1)$, *and*

(ii) M *is Einstein*,

then M is either a complex space form or a Kaehler manifold with vanishing Ricci tensor.

2. **Preliminaries.** Let M be an n-dimensional Kaehler manifold. Let $\omega^1, \dots, \omega^n$ be a local field of unitary frames. Then the Kaehler metric is written as $g = \frac{1}{2}\Sigma(\omega^\alpha \otimes \overline{\omega}^\alpha + \overline{\omega}^\alpha \otimes \omega^\alpha)$ and the fundamental 2-form is given by $\Phi = \frac{1}{2}\sqrt{-1}\,\Sigma\,\omega^\alpha \wedge \overline{\omega}^\alpha$. Let $\Omega^\alpha_\beta = \Sigma R^\alpha_{\beta\gamma\overline{\delta}}\omega^\gamma \wedge \overline{\omega}^\delta$ be the curvature form of M. Then the *curvature tensor* of M is the tensor field with local components $R^\alpha_{\beta\gamma\overline{\delta}}$, which will be denoted by R. The *Ricci tensor* S and the *scalar curvature* ρ are given by

$$S = \frac{1}{2}\sum(R_{\alpha\overline{\beta}}\omega^\alpha \otimes \overline{\omega}^\beta + \overline{R}_{c\overline{\beta}}\overline{\omega}^\alpha \otimes \omega^\beta), \qquad \rho = 2\sum R_{\alpha\overline{\alpha}},$$

where $R_{\alpha\overline{\beta}} = 2\Sigma R^\gamma_{\alpha\gamma\overline{\beta}}$. We denote by $\|R\|$ and $\|S\|$ the length of the curvature tensor and the Ricci tensor, respectively, so that $\|R\|^2 = 16\Sigma R^\alpha_{\beta\gamma\overline{\delta}}R^\beta_{\alpha\delta\overline{\gamma}}$ and $\|S\|^2 = 2\,\Sigma\,R_{\alpha\overline{\beta}}R_{\beta\overline{\alpha}}$.

We prepare the following general result.

Lemma. *Let M be an n-dimensional Kaehler manifold. Then*

Received by the editors August 16, 1974 and, in revised form, October 25, 1974.

AMS (MOS) subject classifications (1970). Primary 53C55; Secondary 57D20.

Key words and phrases. Compact Einstein-Kaehler manifold, complex space form, Chern class.

[1]Work done under partial support by NSF Grant GP-36684.

[2]Work done under partial support by NSF Grant GP-36684 and the Matsunaga Science Foundation.

$$\tfrac{1}{2}n(n+1)\|R\|^2 \geq 2n\|S\|^2 \geq \rho^2.$$

The first equality holds if and only if M is a complex space form, and the second equality holds if and only if M is Einstein.

Proof. The first inequality is obtained by considering the length of the tensor field with local components $R^{\alpha}_{\beta\gamma\bar\delta} - [2(n+1)]^{-1}(\delta^{\alpha}_{\gamma}R_{\beta\bar\delta} + \delta^{\alpha}_{\beta}R_{\gamma\bar\delta})$. It is well known that this tensor field vanishes if and only if M is a complex space form (cf., for example, [2]).

The second inequality is obtained by considering the tensor field with local components $R_{\alpha\bar\beta} - \rho\delta_{\alpha\beta}/2n$. It is clear that this tensor field vanishes if and only if M is Einstein. Q.E.D.

If we define a closed $2k$-form γ_k by

$$\gamma_k = \frac{(-1)^k}{(2\pi\sqrt{-1})^k k!}\sum \delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}\Omega^{\beta_1}_{\alpha_1}\wedge\cdots\wedge\Omega^{\beta_k}_{\alpha_k},$$

then the kth Chern class c_k of M is represented by γ_k. In particular, c_1 and c_2 are represented by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\sum\Omega^{\alpha}_{\alpha} \quad\text{and}\quad \gamma_2 = -\frac{1}{8\pi^2}\sum(\Omega^{\alpha}_{\alpha}\wedge\Omega^{\beta}_{\beta} - \Omega^{\alpha}_{\beta}\wedge\Omega^{\beta}_{\alpha}),$$

respectively.

3. **Proof of Theorem.** It follows from assumption (i) that

$$\gamma_1^{n-2}\gamma_2 = n\gamma_1^n/2(n+1) + d\eta \tag{3}$$

for some $(2n-1)$-form η.

From assumption (ii), we have $S = \rho g/2n$ and, hence,

$$\sum\Omega^{\alpha}_{\alpha} = \frac{\rho}{4n}\sum\omega^{\alpha}\wedge\bar\omega^{\alpha} = \frac{\rho}{2\sqrt{-1}n}\Phi.$$

This, together with (1) and (2), implies that

$$\gamma_1 = \frac{\rho}{4n\pi}\Phi \quad\text{and}\quad \gamma_2 = \frac{\rho^2}{32n^2\pi^2}\Phi^2 + \frac{1}{8\pi^2}\sum\Omega^{\alpha}_{\beta}\wedge\Omega^{\beta}_{\alpha}.$$

Therefore (3) is reduced to

$$\frac{\rho^n}{2(n+1)(4n\pi)^n}\Phi^n + \frac{2n^2\rho^{n-2}}{(4n\pi)^n}\Phi^{n-2}\wedge\left(\sum\Omega^{\alpha}_{\beta}\wedge\Omega^{\beta}_{\alpha}\right) = d\eta. \tag{4}$$

Let Λ be the operator of interior product by Φ. Applying Λ^n to both sides of (4), we may obtain

$$\frac{n!n!\rho^n}{2(n+1)(4n\pi)^n} + \frac{n^2n!(n-2)!\rho^{n-2}}{(4n\pi)^n}(\tfrac{1}{2}\|R\|^2 - \|S\|^2) = \Lambda^n d\eta.$$

Integrating both sides of this equation, we have

$$\int_M \rho^{n-2}\left\{\frac{1}{2(n+1)}\rho^2 + \frac{n}{2(n-1)}\|R\|^2 - \frac{n}{n-1}\|S\|^2\right\}*1 = 0, \tag{5}$$

where $*1$ denotes the volume element of M.

From assumption (ii) and Lemma, $\rho^2 = 2n\|S\|^2$ and, hence, (5) can be written as

$$\rho^{n-2} \int_M \left(\|R\|^2 - \frac{4}{n+1} \|S\|^2 \right) * 1 = 0.$$

This, together with Lemma, implies that either $\rho = 0$ or $\|R\|^2 = 4\|S\|^2/(n+1)$. Therefore, M is either Ricci-flat or of constant holomorphic sectional curvature.

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