COMPLETE DOMAINS WITH RESPECT TO THE CARATHÉODORY DISTANCE. II

DONG S. KIM

ABSTRACT. In [1] we have obtained the following result: Let $D$ be a bounded domain in $\mathbb{C}^n$. Suppose there is a compact subset $K$ of $D$ such that for every $x \in D$ there is an analytic automorphism $f \in \text{Aut}(D)$ and a point $a \in K$ such that $f(x) = a$. Then $D$ is a domain of bounded holomorphy, in the sense that $D$ is the maximal domain on which every bounded holomorphic function on $D$ can be continued holomorphically (cf. Narasimhan [2, Proposition 7, p. 127]). Here we shall give a stronger result: Under the same assumptions, $D$ is $c$-complete. We note that a $c$-complete domain is a domain of bounded holomorphy, in particular, a domain of holomorphy. A domain of bounded holomorphy, however, need not be $c$-complete.

Let $D$ be a bounded domain in $\mathbb{C}^n$. Let $p \in D$ and $q \in \overline{D}$. We define $c(p, q) = \lim_{y \to q} \inf_{y \to p} c(p, y)$, where $c$ is the Carathéodory distance on $D$. A boundary point $q$ of $D$ is called an infinite distance boundary point if there is at least one sequence $(q_n)$ of points of $D$ which converges to $q$ such that $c(p, q_n) \to \infty$ as $n \to \infty$, $p \in D$. This point $q$ is called a stable infinite distance boundary point if $c(p, q_n) \to \infty$ as $n \to \infty$ for every sequence $(q_n) \to q$. We define the minimal boundary distance from $p$ to the boundary of $D$ by $\inf_{q \in \partial D} c(p, q)$. If $p$ is replaced by a compact subset $K$ of $D$, then the minimal boundary distance from $K$ to the boundary of $D$ is given by $\inf_{p \in K} (\inf_{q \in \partial D} c(p, q))$. We denote this by $\min c(K, \partial D)$. We observe that if $D$ has exclusively stable infinite distance boundary points, $\min c(K, \partial D) = \infty$ for every compact subset $K$ of $D$. If $D$ has an unstable infinite distance boundary point $q$ or a finite distance boundary point $q$, then $\min c(K, q) < \infty$ for every compact subset $K$ of $D$.

Theorem. Let $D$ be a bounded domain in $\mathbb{C}^n$. Suppose there is a compact subset $K$ of $D$ such that for any $x \in D$ there is an analytic automorphism $f \in \text{Aut}(D)$ and a point $a \in K$ such that $f(x) = a$. Then $D$ is $c$-complete.

Proof. Assume that $D$ is not $c$-complete. Then there is a boundary point which is not of stably infinite distance. Let $r = \min c(K, \partial D)$, where $K$ is a compact subset of $D$ in the hypothesis. Fix $q \in \partial D$ such that $c(K, q) = r$. Choose a sequence of points $\{x_n\}$ of $D$ such that $\{x_n\} \to q$.
and \( c(x_0, x_n) < r/3 \) for all \( n \). Let \( f_n \in {\text{Aut}}(D) \) such that \( f_n(x_n) = a_n \in K \) for all \( n \). Since \( K \) is compact, \( \{a_n\} \to a \in K \). The family \( \{f_n\} \) of automorphisms of \( D \) is uniformly bounded so that there is a subsequence \( \{f_{k_j}\} \) which converges uniformly on compact subsets of \( D \) to a holomorphic mapping \( f: D \to \overline{D} \). Then we have \( f_{k_j}(x_0) \to f(x_0) \) and

\[
\frac{r}{3} \geq c(x_0, x_{k_j}) = c(f_{k_j}(x_0), f_{k_j}(x_{k_j})) = c(f_{k_j}(x_0), a_{k_j}) \quad \text{for all } k_j.
\]

Since the distance \( c \) is continuous, \( c(f_{k_j}(x_0), a_{k_j}) \to c(f(x_0), a) \). Since \( a \in K \) and \( f(x_0) \in \{x \in \overline{D} : \min c(K, x) \leq r/3\} \subseteq D, f(x_0) \in D \). So \( f(D) \not\subseteq \partial D \). By a theorem of Cartan (see, for instance, Narasimhan [2, Theorem 4, p. 78]), \( f \) is an automorphism of \( D \). But this is absurd since \( f_{k_j}^{-1}(a_{k_j}) = x_{k_j} \), if \( a \) is a limit point of \( \{a_k\} \) in \( K \), \( f^{-1}(a) \in D \). But \( \{x_k\} \) has no limit point in \( D \). Hence \( D \) is \( c \)-complete.

**Corollary.** If \( \Gamma \) is a discrete subgroup of \( {\text{Aut}}(D) \) such that \( D/\Gamma \) is compact, then \( D \) is \( c \)-complete.

**Corollary.** If \( D \) is a bounded homogeneous domain then \( D \) is \( c \)-complete.

**Remark.** We may also claim the last corollary by the following facts.

Since every bounded homogeneous domain in \( \mathbb{C}^n \) is biholomorphic to an affinely homogeneous Siegel domain of second kind, and a Siegel domain of second kind is \( c \)-complete, a bounded homogeneous domain is \( c \)-complete.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611