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A CHARACTERISATION OF LIPSCHITZ CLASSES ON 0-DIMENSIONAL GROUPS

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ABSTRACT. This paper is concerned with characterising, in terms of certain properties of their Fourier transforms, the Lipschitz functions of order α $(0 < \alpha < 1)$ defined on a locally compact metric 0-dimensional Abelian group.

1. **Introduction.** Let G be a locally compact metric 0-dimensional Abelian group with translation-invariant metric d. Its character group will be denoted by Γ.

We shall characterise the Lipschitz functions of order α $(0 < \alpha < 1)$ in terms of certain properties of their Fourier transforms. The results obtained are analogues of classical results due to Jackson and Bernstein (see [5, Chapter 3, Theorems (13.6), (13.20), respectively]).

2. **Notation and preliminary results.** Choose any strictly-decreasing sequence (β_n) of positive numbers smaller than 1 for which there exists $\mu \in (0, 1)$ such that $\beta_{n+1} \leq \mu\beta_n$ for all $n \in \{1, 2, \ldots\}$. Consider the open sets

$$V_n = \{x \in G: d(x, 0) < \beta_n\}.$$

Since $\{V_n\}_{n=1}^{\infty}$ is an open basis at 0, it follows from [3, (7.7)] that all but at most a finite number of the V_n are contained in a compact subgroup of G. By deleting some of the β_n if necessary, we can assume that all the V_n enjoy this property.

Now put $\mathfrak{T}_n = A(\Gamma, V_n)$ (the annihilator of V_n in Γ). Then each $\mathfrak{T}_n$ is a compact open subgroup of Γ, and $\mathfrak{T}_1 \subset \mathfrak{T}_2 \subset \cdots$. Furthermore, since (by [3, (24.18)]) every element of Γ lies in a compact subgroup of Γ, it follows that $\Gamma = \bigcup_{n=1}^{\infty} \mathfrak{T}_n$.

Let λ denote a chosen Haar measure on G. The spectrum (written $\Sigma(f)$) of $f \in L^{\infty}(G)$ will be defined as in [3, (40.21)]. For $f \in L^p(G)$ $(p \in [1, \infty))$, we define its spectrum by

$$\Sigma(f) = \bigcup\{\Sigma(\phi * f): \phi \in C_{00}(G)\}$$

(where $C_{00}(G)$ denotes the space of continuous functions on G with compact support). We shall write

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$$L^p_{\mathfrak{T}}(G) = \{f \in L^p(G) \colon \Sigma(f) \subset \mathfrak{T}\},$$

where $\mathfrak{T}$ is any subset of Γ.

Given $f \in L^p(G)$, its best pth power approximation by members of $L^p_{\mathfrak{T}_n}(G)$ is written as

$$\text{(i)} \qquad E_n(p; f) = \inf\{\|f - t\|_p \colon t \in L^p_{\mathfrak{T}_n}(G)\},$$

and its mean modulus of continuity with exponent p is given by

$$\omega(p; f; \delta) = \sup\{\|\tau_a f - f\|_p \colon d(a, 0) \leq \delta\}$$

(where $\tau_a f \colon x \longrightarrow f(x - a)$). We shall define

$$\mathrm{Lip}_p \alpha = \{f \in L^p(G) \colon \omega(p; f; \delta) = O(\delta^\alpha)\},$$

where $\alpha > 0$. In the case $p = \infty$ it will be further assumed that such f are continuous (and hence uniformly continuous).

Lemma. *For every $f \in L^p(G)$ there exists $t^* \in L^p_{\mathfrak{T}_n}(G)$ such that the infimum in* (i) *is attained.*

Proof. We can choose a sequence $(t_m) \subset L^p_{\mathfrak{T}_n}(G)$ which, for each $m \in \{1, 2, \ldots\}$, satisfies $\|f - t_m\|_p \leq E_n(p; f) + m^{-1}$. Then

$$\|t_m\|_p \leq E_n(p; f) + m^{-1} + \|f\|_p \leq M$$

for some M independent of m. We now consider two cases:

(a) $1 < p \leq \infty$. The ball $\{g \in L^p(G) \colon \|g\|_p \leq M\}$ is weak*-compact in $L^p(G)$, and hence there exists $t^* \in L^p(G)$ and a subsequence $(t_{m(k)})$ of (t_m) for which $\int_G t_{m(k)} h \, d\lambda \longrightarrow \int_G t^* h \, d\lambda$ for all $h \in L^{p'}(G)$ (where p' is the exponent conjugate to p, that is, $p^{-1} + p'^{-1} = 1$ for $p \neq \infty$, and $\infty' = 1$).

Choose any $h \in L^{p'}_{\mathfrak{T}_n^c}(G)$. Then, given $\epsilon > 0$, there exists k_0 such that for all $k \geq k_0$, $|\int_G t_{m(k)} h \, d\lambda - \int_G t^* h \, d\lambda| < \epsilon$. Now $\Sigma(t_{m(k)}) \subset \mathfrak{T}_n$ and $\Sigma(h_\vee) = -\Sigma(h) \subset \mathfrak{T}_n^c$ (where $h_\vee \colon x \longrightarrow h(-x)$); consequently $\Sigma(t_{m(k)} * h_\vee) \subset \Sigma(t_{m(k)}) \cap \Sigma(h_\vee)$ is void (this inclusion follows from [3, (40.21)] and the (easily proved) property of the spectrum that $\Sigma(f) = \bigcup\{\Sigma(\phi * f) \colon \phi \in C_{00}(G)\}$ is valid for all $f \in L^\infty(G)$). Hence

$$\int_G t_{m(k)} h \, d\lambda = t_{m(k)} * h_\vee(0) = 0.$$

Thus $|\int_G t^* h \, d\lambda| < \epsilon$ and, since $\epsilon > 0$ was chosen arbitrarily,

$$t^* * h_\vee(0) = \left| \int_G t^* h \, d\lambda \right| = 0.$$

Now this holds for all $h \in L^{p'}_{\mathfrak{T}_n^c}(G)$, whence it follows that $\Sigma(t^*) \subset \mathfrak{T}_n$.

We now show that t^* fulfills the requirement of the Lemma. For $h \in L^{p'}(G)$, $\|h\|_{p'} \leq 1$, and $\epsilon > 0$ given we can find k_0 such that

$$\max\left\{m(k_0)^{-1}, \left|\int_G t_{m(k_0)} h\, d\lambda - \int_G t^* h\, d\lambda\right|\right\} \leq \epsilon.$$

Then

$$\begin{aligned}\left|\int_G fh\, d\lambda - \int_G t^* h\, d\lambda\right| &\leq \left|\int_G fh\, d\lambda - \int_G t_{m(k_0)} h\, d\lambda\right| \\ &\quad + \left|\int_G t_{m(k_0)} h\, d\lambda - \int_G t^* h\, d\lambda\right| \\ &\leq \|f - t_{m(k_0)}\|_p + \epsilon \leq E_n(p; f) + 2\epsilon.\end{aligned}$$

As this holds for all $\epsilon > 0$, we must have

$$\left|\int_G fh\, d\lambda - \int_G t^* h\, d\lambda\right| \leq E_n(p; f),$$

and

$$\|f - t^*\|_p = \sup\left\{\left|\int_G fh\, d\lambda - \int_G t^* h\, d\lambda\right| : \|h\|_{p'} \leq 1\right\} \leq E_n(p; f).$$

The reverse inequality is trivial, thus taking care of (a).

(b) $p = 1$. We consider $L^1(G)$ embedded as a subspace in $M_b(G)$, the space of bounded Radon measures on G with the topology as the dual of $C_0(G)$ (the space of continuous functions on G that vanish at infinity). Denoting by μ_g the measure generated by $g \in L^1(G)$, we have that the ball $\{\mu_g \in M_b(G): \|\mu_g\| \leq M\}$ is weak*-compact in $M_b(G)$, and hence there exists $\mu \in M_b(G)$ and a subsequence $(t_{m(k)})$ of (t_m) for which $\mu_{t_{m(k)}}(h) \longrightarrow \mu(h)$ for all $h \in C_0(G)$. An argument similar to that in the second paragraph of (a) gives that $\operatorname{supp}(\hat{\mu}) \subset \mathbf{T}_n$.

Now, from [3, (31.7) and (23.10)], $\hat{\mu} = \hat{\mu}\hat{k}_n$, where $k_n = \lambda(V_n)^{-1}\xi_{V_n}$; we deduce from the fact that the Fourier transform is one-to-one that $\mu = \mu_t$ for some $t \in L^1(G)$. The remainder of the proof now follows as in (a).

3. **Characterisations of $\operatorname{Lip}_p\alpha$.** The first result we require is the following analogue of Bernstein's theorem:

Theorem 1. *If, for some* $\alpha > 0$, $E_n(p; f) = O(\beta_{n+1}^{\alpha})$, *then*

$$\omega(p; f; \delta) = \begin{cases} O(\delta^{\alpha}), & 0 < \alpha < 1, \\ O(\delta|\log \delta|), & \alpha = 1, \\ O(\delta), & \alpha > 1. \end{cases}$$

Proof. For each $n \in \{1, 2, \ldots\}$ we have (see the preceding Lemma) the existence of $t_n^* \in L^p_{\mathbf{T}_n}(G)$ for which $E_n(p; f) = \|f - t_n^*\|_p$. By assumption

$\|f - t_n^*\|_p \leq B\beta_{n+1}^\alpha$ for some constant $B > 0$. Define

$$s_1 = t_1^*; \quad s_n = t_n^* - t_{n-1}^* \qquad (n \in \{2, 3, \ldots\}).$$

Then

$$\|s_n\|_p \leq \|t_n^* - f\|_p + \|f - t_{n-1}^*\|_p \leq B(\beta_{n+1}^\alpha + \beta_n^\alpha) \leq 2B\beta_n^\alpha \quad (n \in \{2, 3, \ldots\}).$$

Thus we can find $C > 0$ such that for all $n \in \{1, 2, \ldots\}$,

$$\text{(ii)} \qquad \|s_n\|_p \leq C\beta_n^\alpha.$$

Now $\sum_{k=1}^n s_k = t_n^*$ converges in $L^p(G)$ to f as $n \to \infty$. Hence, for any $a \in G$, $\tau_a(\sum_{k=1}^n s_k) - \sum_{k=1}^n s_k$ converges in $L^p(G)$ to $\tau_a f - f$ as $n \to \infty$, and

$$\text{(iii)} \qquad \begin{aligned} \|\tau_a f - f\|_p &\leq \sum_{k=1}^\infty \|\tau_a s_k - s_k\|_p \leq \sum_{k=1}^m \|\tau_a s_k - s_k\|_p + 2\sum_{k=m+1}^\infty \|s_k\|_p \\ &\leq 2\sum_{k=1}^m \omega_{\mathbf{T}_k}(a)\|s_k\|_p + 2\sum_{k=m+1}^\infty \|s_k\|_p; \end{aligned}$$

to make the last step we use [1, Theorem 1.3], which asserts that for $s \in L_{\mathbf{T}}^p(G)$ with $\mathbf{T}$ compact, and for a relatively compact open set $\nabla \subset \Gamma$ containing zero, we have

$$\|\tau_a s - s\|_p \leq (\theta(\mathbf{T} + \nabla - \nabla)/\theta(\mathbf{T}))^{1/2}(\omega_\nabla(a) + \omega_{\mathbf{T}+\nabla-\nabla}(a))\|s\|_p,$$

where θ denotes a Haar measure on Γ chosen so that Plancherel's theorem holds, and $\omega_{\mathbf{T}}(a) = \sup\{|\gamma(a) - 1| : \gamma \in \mathbf{T}\}$ (note that in (iii), $\mathbf{T}_k$ is a compact open subgroup of Γ). As $d(a, 0) \geq \beta_k$ for all $a \notin V_k$, we have for such a

$$\text{(iv)} \qquad \omega_{\mathbf{T}_k}(a) \leq 2\beta_k^{-1}d(a, 0).$$

But $\omega_{\mathbf{T}_k}(a) = 0$ for $a \in V_k$ and so (iv) clearly holds for all $a \in G$. This, combined with (ii) and (iii), gives

$$\omega(p; f; \delta) \leq 4C\delta \sum_{k=1}^m \beta_k^{\alpha-1} + 2C\sum_{k=m+1}^\infty \beta_k^\alpha$$

for any $\delta > 0$.

Now suppose that $0 < \delta \leq \beta_1$, and choose $m \geq 1$ so that $\beta_{m+1} < \delta \leq \beta_m$. Then

$$\begin{aligned} \omega(p; f; \delta) &\leq 4C\delta \sum_{k=1}^m \beta_k^{\alpha-1} + 2C\delta^\alpha \sum_{k=m+1}^\infty \left(\frac{\beta_k}{\beta_{m+1}}\right)^\alpha \\ &\leq 4C\delta \sum_{k=1}^m \beta_k^{\alpha-1} + \frac{2C}{1-\mu^\alpha}\delta^\alpha \end{aligned}$$

(recall that $0 < \mu < 1$ and $\beta_{n+1} \leq \mu\beta_n$ for all $n \in \{1, 2, \ldots\}$). We now consider three cases according to whether α is smaller than, equal to, or greater than 1.

(a) $0 < \alpha < 1$. Since $\beta_{n+1} \leq \mu\beta_n$ for all $n \in \{1, 2, \ldots\}$, we have

$$\sum_{k=1}^{m} \beta_k^{\alpha-1} \leq \sum_{k=1}^{m} \mu^{(1-\alpha)(m-k)}\beta_m^{\alpha-1} \leq \frac{1}{1-\mu^{1-\alpha}}\delta^{\alpha-1}.$$

Thus

$$\omega(p; f; \delta) \leq \frac{4C}{1-\mu^{1-\alpha}}\delta^{\alpha} + \frac{2C}{1-\mu^{\alpha}}\delta^{\alpha} = O(\delta^{\alpha}).$$

(b) $\alpha = 1$. Now

$$\omega(p; f; \delta) \leq 4C\delta m + 2C\delta/(1-\mu).$$

But $\delta \leq \beta_m \leq \mu^{m-1}\beta_1 < 1$ implies that

$$m \leq (|\log \delta| + |\log \beta_1|)/|\log \mu| + 1,$$

whence the result follows.

(c) $\alpha > 1$. We have

$$\omega(p; f; \delta) \leq \left(4C\sum_{k=1}^{m} \beta_k^{\alpha-1} + \frac{2C}{1-\mu^{\alpha}}\delta^{\alpha-1}\right)\delta = O(\delta),$$

and the theorem is proved.

Theorem 1, together with the property $\Gamma = \bigcup_{n=1}^{\infty} \mathfrak{T}_n$, guarantees us a rich supply of functions in $\mathrm{Lip}_p\alpha$ (at least in the case when G is infinite).

If we have that each V_n is actually a (compact open) subgroup of G, then the remarks in [2, p. 64] show that

$$E_n(p; f) \leq \sup\{\|\tau_a f - f\|_p : a \in V_n\}$$

for all $f \in L^p(G)$ (or, if $p = \infty$, for uniformly continuous f). In particular, if $f \in \mathrm{Lip}_p\alpha$ for some $\alpha > 0$ and $\beta_{n+1} = \mu\beta_n$ for all $n \in \{1, 2, \ldots\}$, then $E_n(p; f) = O(\beta_{n+1}^{\alpha})$. This result combines with Theorem 1 to give (with β_n, V_n as above)

Theorem 2. *Let $\alpha \in (0, 1)$ be given. Then $f \in \mathrm{Lip}_p\alpha$ if and only if $E_n(p; f) = O(\beta_{n+1}^{\alpha})$ (where, for $p = \infty$, f is taken to be uniformly continuous).*

We can obtain more from [2, p. 64], namely that

$$S_n(p; f) \leq \sup\{\|\tau_a f - f\|_p : a \in V_n\},$$

where $S_n(p; f) = \|k_n * f - f\|_p$, $k_n = \lambda(V_n)^{-1}\xi_{V_n}$ and ξ_{V_n} denotes the characteristic function of V_n. We then have (with β_n, V_n as in Theorem 2)

Theorem 3. *Let $\alpha \in (0, 1)$ be given. Then $f \in \mathrm{Lip}_p\alpha$ if and only if $S_n(p; f) = O(\beta_{n+1}^{\alpha})$ (where, for $p = \infty$, f is taken to be uniformly continuous).*

Note that if G is taken to be compact in Theorem 3, then each $k_n * f$ is just a partial sum of the Fourier series of f.

Theorems 2 and 3 require that each V_n be a subgroup of G. This happens, for example, if we have a neighbourhood basis (V_n) at zero consisting of a strictly-decreasing sequence of compact open subgroups of G (such a neighbourhood basis always exists in an infinite first-countable 0-dimensional locally compact group; see [3, (7.7)]), any strictly-decreasing sequence (β_n) of positive numbers, and a translation-invariant metric d defined by

$$d(x, y) = \begin{cases} \beta_{n+1}, & x - y \in V_n \setminus V_{n+1}, \\ \beta_1, & x - y \notin V_1, \\ 0, & x = y, \end{cases}$$

(cf. [4, §2]). In this case Theorems 2 and 3 are valid provided that there exists $\mu \in (0, 1)$ such that $\beta_{n+1} \leq \mu\beta_n$ for all $n \in \{1, 2, \ldots\}$.

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