A CHARACTERISATION OF LIPSCHITZ CLASSES
ON 0-DIMENSIONAL GROUPS

WALTER R. BLOOM

ABSTRACT. This paper is concerned with characterising, in terms of certain properties of their Fourier transforms, the Lipschitz functions of order \( \alpha \) \((0 < \alpha < 1)\) defined on a locally compact metric 0-dimensional Abelian group.

1. Introduction. Let \( G \) be a locally compact metric 0-dimensional Abelian group with translation-invariant metric \( d \). Its character group will be denoted by \( \Gamma \).

We shall characterise the Lipschitz functions of order \( \alpha \) \((0 < \alpha < 1)\) in terms of certain properties of their Fourier transforms. The results obtained are analogues of classical results due to Jackson and Bernstein (see [5, Chapter 3, Theorems (13.6), (13.20), respectively]).

2. Notation and preliminary results. Choose any strictly-decreasing sequence \( (\beta_n) \) of positive numbers smaller than 1 for which there exists \( \mu \in (0, 1) \) such that \( \beta_{n+1} \leq \mu \beta_n \) for all \( n \in \{1, 2, \ldots\} \). Consider the open sets

\[ V_n = \{ x \in G : d(x, 0) < \beta_n \} \]

Since \( \{V_n\}_{n=1}^{\infty} \) is an open basis at 0, it follows from [3, (7.7)] that all but at most a finite number of the \( V_n \) are contained in a compact subgroup of \( G \). By deleting some of the \( \beta_n \) if necessary, we can assume that all the \( V_n \) enjoy this property.

Now put \( T_n = \mathcal{A}(\Gamma, V_n) \) (the annihilator of \( V_n \) in \( \Gamma \)). Then each \( T_n \) is a compact open subgroup of \( \Gamma \), and \( T_1 \subset T_2 \subset \cdots \). Furthermore, since (by [3, (24.18)]) every element of \( \Gamma \) lies in a compact subgroup of \( \Gamma \), it follows that \( \Gamma = \bigcup_{n=1}^{\infty} T_n \).

Let \( \lambda \) denote a chosen Haar measure on \( G \). The spectrum (written \( \Sigma(\phi) \)) of \( f \in L^\infty(G) \) will be defined as in [3, (40.21)]. For \( f \in L^p(G) \) \((p \in [1, \infty])\), we define its spectrum by

\[ \Sigma(f) = \bigcup \{ \Sigma(\phi * f) : \phi \in C_{00}(G) \} \]

(where \( C_{00}(G) \) denotes the space of continuous functions on \( G \) with compact support). We shall write
where \( T \) is any subset of \( \Gamma \).

Given \( f \in L^p(G) \), its best \( p \)th power approximation by members of \( L^p_{T_n}(G) \) is written as

\[
E_n(p; f) = \inf \{ \| f - t \|_p : t \in L^p_{T_n}(G) \},
\]

and its mean modulus of continuity with exponent \( p \) is given by

\[
\omega(p; f; \delta) = \sup \{ \| \tau_{a} f - f \|_p : d(a, 0) \leq \delta \}
\]

(where \( \tau_{a} f : x \rightarrow f(x - a) \)). We shall define

\[
\text{Lip}^p_{\alpha} = \{ f \in L^p(G) : \omega(p; f; \delta) = O(\delta^\alpha) \},
\]

where \( \alpha > 0 \). In the case \( p = \infty \) it will be further assumed that such \( f \) are continuous (and hence uniformly continuous).

**Lemma.** For every \( f \in L^p(G) \) there exists \( t^* \in L^p_{T_n}(G) \) such that the infimum in (i) is attained.

**Proof.** We can choose a sequence \( (t_m) \subset L^p_{T_n}(G) \) which, for each \( m \in \{1, 2, \ldots \} \), satisfies \( \| f - t_m \|_p \leq E_n(p; f) + m^{-1} \). Then

\[
\| t_m \|_p \leq E_n(p; f) + m^{-1} + \| f \|_p \leq M
\]

for some \( M \) independent of \( m \). We now consider two cases:

(a) \( 1 < p < \infty \). The ball \( \{ g \in L^p(G) : \| g \|_p \leq M \} \) is weak* compact in \( L^p(G) \), and hence there exists \( t^* \in L^p(G) \) and a subsequence \( (t_{m(k)}) \) of \( (t_m) \) for which \( \int_G t_{m(k)} h d\lambda \rightarrow \int_G t^* h d\lambda \) for all \( h \in L^{p'}(G) \) (where \( p' \) is the exponent conjugate to \( p \), that is, \( p^{-1} + p'^{-1} = 1 \) for \( p \neq \infty \), and \( \infty' = 1 \)).

Choose any \( h \in L^{p'}_{T_n}(G) \). Then, given \( \epsilon > 0 \), there exists \( k_0 \) such that for all \( k \geq k_0 \), \( \int_G | t_{m(k)} h - t^* h | d\lambda < \epsilon \). Now \( \Sigma(t_{m(k)}) \subset T_n \) and \( \Sigma(h \vee) = -\Sigma(h) \subset T_n' \) (where \( h \vee : x \rightarrow \max\{0, x\} \)); consequently \( \Sigma(t_{m(k)} * h \vee) \subset \Sigma(t_{m(k)}) \cap \Sigma(h \vee) \) is void (this inclusion follows from [3, (40.21)] and the (easily proved) property of the spectrum that \( \Sigma(f) = \bigcup \{ \Sigma(\phi * f) : \phi \in C_{00}(G) \} \) is valid for all \( f \in L^\infty(G) \)). Hence

\[
\int_G t_{m(k)} h d\lambda = t_{m(k)} * h \vee(0) = 0.
\]

Thus \( \int_G | t^* h | d\lambda < \epsilon \) and, since \( \epsilon > 0 \) was chosen arbitrarily,

\[
t^* * h \vee(0) = \int_G t^* h d\lambda = 0.
\]

Now this holds for all \( h \in L^{p'}_{T_n}(G) \), whence it follows that \( \Sigma(t^*) \subset T_n' \).
We now show that \( t^* \) fulfills the requirement of the Lemma. For \( h \in L^p(G) \), \( \|h\|_p \leq 1 \), and \( \epsilon > 0 \) given we can find \( k_0 \) such that

\[
\max \left\{ m(k_0)^{-1}, \left| \int_G t m(k_0) h \, d\lambda - \int_G t^* h \, d\lambda \right| \right\} \leq \epsilon.
\]

Then

\[
\left| \int_G h \, d\lambda - \int_G t^* h \, d\lambda \right| \leq \left| \int_G h \, d\lambda - \int_G t m(k_0) h \, d\lambda \right|
+ \left| \int_G t m(k_0) h \, d\lambda - \int_G t^* h \, d\lambda \right|
\leq \|h - t m(k_0)\|_p + \epsilon \leq E_n(p; f) + 2\epsilon.
\]

As this holds for all \( \epsilon > 0 \), we must have

\[
\left| \int_G h \, d\lambda - \int_G t^* h \, d\lambda \right| \leq E_n(p; f),
\]

and

\[
\|f - t^*\|_p = \sup \left\{ \left| \int_G h \, d\lambda - \int_G t^* h \, d\lambda \right| : \|h\|_p \leq 1 \right\} \leq E_n(p; f).
\]

The reverse inequality is trivial, thus taking care of (a).

(b) \( p = 1 \). We consider \( L^1(G) \) embedded as a subspace in \( M_b(G) \), the space of bounded Radon measures on \( G \) with the topology as the dual of \( C_0(G) \) (the space of continuous functions on \( G \) that vanish at infinity). Denoting by \( \mu_g \) the measure generated by \( g \in L^1(G) \), we have that the ball \( \{ \mu_g \in M_b(G) : \|\mu_g\| \leq M \} \) is weak*-compact in \( M_b(G) \), and hence there exists \( \mu \in M_b(G) \) and a subsequence \( (t_{m(k)}) \) of \( (t_m) \) for which \( \mu_{m(k)}(h) \to \mu(h) \) for all \( h \in C_0(G) \). An argument similar to that in the second paragraph of (a) gives that \( \text{supp}(\hat{\mu}) \subset \mathbb{T}_n \).

Now, from [3, (31.7) and (23.10)], \( \hat{\rho} = \hat{\mu} \hat{k}_{n} \), where \( k_n = \lambda(V_n)^{-1} \xi_{V_n} \); we deduce from the fact that the Fourier transform is one-to-one that \( \mu = \mu_t \) for some \( t \in L^1(G) \). The remainder of the proof now follows as in (a).

3. Characterisations of Lip\(_\alpha\). The first result we require is the following analogue of Bernstein's theorem:

**Theorem 1.** If, for some \( \alpha > 0 \), \( E_n(p; f) = O(\beta_n^{\alpha+1}) \), then

\[
\omega(p; f; \delta) = \begin{cases} 
O(\delta^\alpha), & 0 < \alpha < 1, \\
O(\delta |\log \delta|), & \alpha = 1, \\
O(\delta), & \alpha > 1.
\end{cases}
\]

**Proof.** For each \( n \in \{1, 2, \ldots\} \) we have (see the preceding Lemma) the existence of \( t_n^* \in L^p_T(G) \) for which \( E_n(p; f) = \|f - t_n^*\|_p \). By assumption
\[ \|f - t_n^*\|_p \leq B \beta_n^{\alpha} + B \beta_{n+1}^{\alpha} \leq 2B \beta_n^{\alpha} \quad (n \in \{1, 2, \ldots\}). \]

Then
\[ \|s_n\|_p \leq \|t_n - f\|_p + \|f - t_{n-1}^*\|_p \leq B(\beta_n^{\alpha} + \beta_{n+1}^{\alpha}) \leq 2B \beta_n^{\alpha} \quad (n \in \{1, 2, \ldots\}). \]

Thus we can find \( C > 0 \) such that for all \( n \in \{1, 2, \ldots\}, \)
\[ \|s_n\|_p \leq C \beta_n^{\alpha}. \]

Now \( \sum_{k=1}^{n} s_k = t_n^* \) converges in \( L^p(G) \) to \( f \) as \( n \to \infty \). Hence, for any \( a \in G \), \( \tau_a(\sum_{k=1}^{n} s_k) - \sum_{k=1}^{n} \tau_a s_k \) converges in \( L^p(G) \) to \( \tau_a f - f \) as \( n \to \infty \), and
\[ \|\tau_a f - f\|_p \leq \sum_{k=1}^{m} \|\tau_a s_k - s_k\|_p \leq \sum_{k=1}^{m} \|\tau_a s_k - s_k\|_p + 2 \sum_{k=m+1}^{\infty} \|s_k\|_p \]

(iii)
\[ \leq 2 \sum_{k=1}^{m} \omega_{T_k}(a)\|s_k\|_p + 2 \sum_{k=m+1}^{\infty} \|s_k\|_p, \]

to make the last step we use [1, Theorem 1.3], which asserts that for \( s \in L^p(G) \) with \( T \) compact, and for a relatively compact open set \( \nabla \subset \Gamma \) containing zero, we have
\[ \|\tau_a s - s\|_p \leq (\theta(T + \nabla - \nabla)/\theta(T))^{1/2} (\omega_\nabla(a) + \omega_{T+\nabla-\nabla}(a))\|s\|_p, \]

where \( \theta \) denotes a Haar measure on \( \Gamma \) chosen so that Plancherel's theorem holds, and \( \omega_T(a) = \sup \{|\gamma(a) - 1| : \gamma \in T\} \) (note that in (iii), \( T_k \) is a compact open subgroup of \( \Gamma \)). As \( d(a, 0) \geq \beta_k \) for all \( a \notin V_k \), we have for such \( a \)
\[ \omega_{T_k}(a) \leq 2 \beta_k^{-1} d(a, 0). \]

But \( \omega_{T_k}(a) = 0 \) for \( a \in V_k \) and so (iv) clearly holds for all \( a \in G \). This, combined with (ii) and (iii), gives
\[ \omega(p; f; \delta) \leq 4C \delta \sum_{k=1}^{m} \beta_k^{\alpha-1} + 2C \sum_{k=m+1}^{\infty} \beta_k^{\alpha}, \]

for any \( \delta > 0 \).

Now suppose that \( 0 < \delta \leq \beta_1 \), and choose \( m \geq 1 \) so that \( \beta_{m+1} < \delta \leq \beta_m \). Then
\[ \omega(p; f; \delta) \leq 4C \delta \sum_{k=1}^{m} \beta_k^{\alpha-1} + 2C \delta^{\alpha} \sum_{k=m+1}^{\infty} \left( \frac{\beta_k}{\beta_{m+1}} \right)^{\alpha} \]
\[ \leq 4C \delta \sum_{k=1}^{m} \beta_k^{\alpha-1} + \frac{2C}{1 - \mu} \delta^{\alpha}. \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(recall that $0 < \mu < 1$ and $\beta_{n+1} \leq \mu \beta_n$ for all $n \in \{1, 2, \ldots\}$). We now consider three cases according to whether $\alpha$ is smaller than, equal to, or greater than 1.

(a) $0 < \alpha < 1$. Since $\beta_{n+1} \leq \mu \beta_n$ for all $n \in \{1, 2, \ldots\}$, we have
\[
\sum_{k=1}^{m} \beta_k^{\alpha-1} \leq \sum_{k=1}^{m} \mu^{(1-\alpha)(m-k)} \beta_m^{\alpha-1} \leq \frac{1}{1 - \mu^{1-\alpha}} \delta^{\alpha-1}.
\]
Thus
\[
\omega(p; f; \delta) \leq \frac{4C}{1 - \mu^{1-\alpha}} \delta^\alpha + \frac{2C}{1 - \mu^\alpha} \delta^\alpha = O(\delta^\alpha).
\]

(b) $\alpha = 1$. Now
\[
\omega(p; f; \delta) \leq 4C \delta m + 2C \delta/(1 - \mu).
\]
But $\delta \leq \beta_m \leq \mu^{m-1} \beta_1 < 1$ implies that
\[
m \leq (|\log \delta| + |\log \beta_1|)/|\log \mu| + 1,
\]
whence the result follows.

(c) $\alpha > 1$. We have
\[
\omega(p; f; \delta) \leq \left(4C \sum_{k=1}^{m} \beta_k^{\alpha-1} + \frac{2C}{1 - \mu^\alpha} \delta^{\alpha-1}\right) \delta = O(\delta),
\]
and the theorem is proved.

Theorem 1, together with the property $\Gamma = \bigcup_{n=1}^\infty V_n$, guarantees us a rich supply of functions in Lip$_p \alpha$ (at least in the case when $G$ is infinite).

If we have that each $V_n$ is actually a (compact open) subgroup of $G$, then the remarks in [2, p. 64] show that
\[
E_n(p; f) \leq \sup \{\|r_a f - f\|_p : a \in V_n\}
\]
for all $f \in L^p(G)$ (or, if $p = \infty$, for uniformly continuous $f$). In particular, if $f \in$ Lip$_p \alpha$ for some $\alpha > 0$ and $\beta_{n+1} = \mu \beta_n$ for all $n \in \{1, 2, \ldots\}$, then $E_n(p; f) = O(\beta_{n+1}^\alpha)$. This result combines with Theorem 1 to give (with $\beta_n$, $V_n$ as above)

**Theorem 2.** Let $\alpha \in (0, 1)$ be given. Then $f \in$ Lip$_p \alpha$ if and only if $E_n(p; f) = O(\beta_{n+1}^\alpha)$ (where, for $p = \infty$, $f$ is taken to be uniformly continuous).

We can obtain more from [2, p. 64], namely that
\[
F_n(p; f) \leq \sup \{\|r_a f - f\|_p : a \in V_n\},
\]
where $S_n(p; f) = \|k_n * f - f\|_p$, $k_n = \lambda(V_n)^{-1/2} \xi V_n$ and $\xi V_n$ denotes the characteristic function of $V_n$. We then have (with $\beta_n$, $V_n$ as in Theorem 2)

**Theorem 3.** Let $\alpha \in (0, 1)$ be given. Then $f \in \text{Lip}_p \alpha$ if and only if $S_n(p; f) = O(\beta_n^{\alpha+1})$ (where, for $p = \infty$, $f$ is taken to be uniformly continuous).

Note that if $G$ is taken to be compact in Theorem 3, then each $k_n * f$ is just a partial sum of the Fourier series of $f$.

Theorems 2 and 3 require that each $V_n$ be a subgroup of $G$. This happens, for example, if we have a neighbourhood basis $(V_n)$ at zero consisting of a strictly-decreasing sequence of compact open subgroups of $G$ (such a neighbourhood basis always exists in an infinite first-countable $0$-dimensional locally compact group; see [3, (7.7)]), any strictly-decreasing sequence $(\beta_n)$ of positive numbers, and a translation-invariant metric $d$ defined by

$$d(x, y) = \begin{cases} 
\beta_n^{\alpha+1}, & x - y \in V_n \setminus V_n^{\alpha+1}, \\
\beta_1, & x - y \notin V_1, \\
0, & x = y,
\end{cases}$$

(cf. [4, §2]). In this case Theorems 2 and 3 are valid provided that there exists $\mu \in (0, 1)$ such that $\beta_n^{\alpha+1} \leq \mu \beta_n$ for all $n \in \{1, 2, \ldots\}$.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TASMANIA, HOBART, TASMANIA, AUSTRALIA

Current address: School of Mathematical and Physical Sciences, Murdoch University, Murdoch, Western Australia, Australia