SHORTER NOTES

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THE RUDIN-CARLESON THEOREM
FOR VECTOR-VALUED FUNCTIONS
J. GLOBEVIČNIK

ABSTRACT. The following generalization of the Rudin-Carleson theorem is proved. Let $X$ be a complex Banach space and let $f: F \to X$ be a continuous function, where $F$ is a closed subset of the unit circle in $C$ of Lebesgue measure zero. There exists a continuous function $g$ from the closed unit disc to $X$ which is analytic on the open unit disc and satisfies (i) $g|F = f$, (ii) $\max_{|z| \leq 1} \|g(z)\| = \max_{z \in F} \|f(z)\|$.

Throughout, $\Lambda$ (resp. $\bar{\Lambda}$) is the open (resp. closed) unit disc in $C$. Given a complex Banach space $X$ and a compact space $K$, we denote by $C(K, X)$ the (Banach) space of all continuous functions from $K$ to $X$ with sup norm. We denote by $A(\Lambda, X)$ the (Banach) space of all continuous functions from $\Lambda$ to $X$ which are analytic on $\Lambda$, with sup norm. We write $C(K), A(\Lambda)$ for $C(K, C), A(A, C)$, respectively.

We prove the following generalization of the well-known Rudin-Carleson theorem (see [2], [3], [6]).

**Theorem.** Let $X$ be a complex Banach space and $F$ a closed set of Lebesgue measure zero on the unit circle in $C$. Let $f \in C(F, X)$. There exists $g \in A(\Lambda, X)$ such that $g|F = f$ and $\|g\|_{A(\Lambda, X)} = \|f\|_{C(F, X)}$.

Let $K$ be a compact space and let $\mathcal{U}$ be a closed subalgebra of $C(K)$. A set $F \subset K$ is called a peak set for $\mathcal{U}$ if there exists $g \in \mathcal{U}$ such that $g(z) = 1$ ($z \in F$) and $|g(z)| < 1$ ($z \in K \setminus F$). Let $X$ be a complex Banach space, $A$ a closed subspace of $C(K, X)$ and $F$ a closed subset of $K$. Define $kF = \{f \in A : \|f\| = 0\}$. $kF$ is a closed subspace of $A$, hence $A/kF$ with norm $\|f + kF\|$ is a Banach space. It is easy to see that $T$, defined by $T(f|F) = f + kF$, is a one-to-one linear operator from $A|F = 1/|F$: $f \in A|$ onto $A/kF$ (see [5, p. 163]).

**Lemma.** Let $X$ be a complex Banach space, $K$ a compact space, $A$ a closed subspace of $C(K, X)$, and $\mathcal{U}$ a closed subalgebra of $C(K)$. Denote by
The set of all functions of the form \( z \mapsto \phi(z)f(z) \) where \( \phi \in \mathcal{A} \) and \( f \in \mathcal{A} \) and suppose that \( \mathcal{A} \subset \mathcal{A} \). Let \( F \subset K \) be a peak set for \( \mathcal{A} \). Then

(i) \( T: f \mapsto f + kF \) is an isometry from \( \mathcal{A}|F \) onto \( \mathcal{A}/kF \) and, consequently, \( \mathcal{A}|F \) is closed in \( C(F, X) \);

(ii) for each \( f \in \mathcal{A}|F \) there exists \( g \in \mathcal{A} \) satisfying \( g|F = f \) and \( \|g\|_{C(K, X)} = \|f\|_{C(F, X)} \).

To prove (i) and (ii) observe that the corresponding proofs for scalar-valued functions (\cite[p. 163, Theorem 3(c)], \cite[p. 164, Theorem 4]{5}) work for vector-valued functions as well.

**Proof of theorem.** \( A(\Delta) \) is a closed subalgebra of \( C(\overline{\Delta}) \) and it is easy to see that \( A(\Delta, X) \) is a closed subspace of \( C(\overline{\Delta}, X) \). Given \( \phi \in A(\Delta) \) and \( f \in A(\Delta, X) \), it is also easy to see that the function \( z \mapsto \phi(z)f(z) \) is again continuous on \( \overline{\Delta} \) and analytic on \( \Delta \), hence \( A(\Delta)A(\Delta, X) \subset A(\Delta, X) \). By \cite[p. 81]{3} \( F \) is a peak set for \( A(\Delta) \), and by (i) of the Lemma it follows that \( A(\Delta, X)|F \) is closed in \( C(F, X) \). Further, by the Mergelyan theorem for vector-valued functions (see \cite{1}), every function in \( C(F, X) \) is the uniform limit of a sequence of polynomials which means that \( A(\Delta, X)|F \) is dense in \( C(F, X) \). By the preceding discussion it follows that \( A(\Delta, X)|F = C(F, X) \). Now the assertion follows by (ii) of the Lemma. Q.E.D.

**Remark.** The generalization of the Rudin-Carleson theorem to vector-valued functions was motivated by the following problem posed by D. Patil at the Conference on Infinite Dimensional Holomorphy, University of Kentucky, May 1973. Let \( X \) be a complex separable Banach space. Does there exist an analytic function \( f: \Delta \to X \) such that the convex hull of \( f(\Delta) \) is contained and dense in the unit ball of \( X \)? Patil's problem will be discussed in a separate paper. Note that the Theorem gives a solution to this problem in the case when \( X \) is finite dimensional. To see this, let \( F \) be a Cantor set of Lebesgue measure zero on the unit circle in \( C \). Since every compact metric space is a continuous image of \( F \) (see \cite[p. 166]{4}), there exists a continuous function \( f \) from \( F \) onto the closed unit ball \( B \) of a finite-dimensional \( X \) by the compactness of \( B \). Then the extension \( g \in A(\Delta, X) \) given by the Theorem has the property that \( g(\Delta) \) is contained and dense in \( B \).

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**Added in proof.** When the present paper was already in print the author found that a more general theorem than the theorem above was proved by E. L. Stout, *On some restriction algebras* (pp. 6–11 of *Function algebras*, Scott-Foresman, Chicago, Ill., 1966. MR 35 #3447).
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INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, LJUBLJANA, YUGOSLAVIA