A COMMON FIXED-POINT THEOREM FOR COMPACT
CONVEX SEMIGROUPS OF NONEXPANSIVE MAPPINGS
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ABSTRACT. Let $C$ be a bounded closed convex subset of a strictly
convex Banach space and let $S$ be a semigroup of nonexpansive self-map-
pings of $C$ which is convex and compact in the topology of weak point-
wise convergence. If $S$ has the property that $\overline{\mathcal{R}(s_1)} \cap \overline{\mathcal{R}(s_2)} \neq \emptyset$
whenever $s_1, s_2 \in S$, then $S$ has a common fixed point and $F(S)$ is a
nonexpansive retract of $C$.

Throughout this paper, $C$ denotes a bounded closed convex subset of
a (real or complex) Banach space $X$. A family $S$ of mappings $s: C \to C$ is a
semigroup if it is closed under composition; $S$ is convex if it is convex in
the vector space $X^C$ (with the usual pointwise operations). By a common
fixed point of $S$ we mean a point $x$ such that $s(x) = x$ for all $s$ in $S$; the
set of common fixed points is denoted by $F(S)$. We give $X^C$ the product
topology after giving $X$ its weak topology, so that compactness of $S$ refers
to its compactness in the topology of weak pointwise convergence. We say
that $S$ satisfies (FP), (F), (D+), (D), or (I), according to whether the fol-
lowing hold for every pair $s_1, s_2$ in $S$:

\begin{itemize}
  \item [(FP)] $S$ has a common fixed point;
  \item [(F)] $s_1$ and $s_2$ have a common fixed point;
  \item [(D+)] $\overline{\mathcal{R}(s_1)} \cap \overline{\mathcal{R}(s_2)} \neq \emptyset$;
  \item [(D)] $\text{dis} (\overline{\mathcal{R}(s_1)}, \overline{\mathcal{R}(s_2)}) = 0$;
  \item [(I)] $\overline{\mathcal{R}(s_1)} \cap \overline{\mathcal{R}(s_2)} \neq \emptyset$,
\end{itemize}

where $\mathcal{R}(s)$ denotes the range of $s$ and $\overline{\mathcal{R}}$ denotes convex closure. Evi-
dently (FP) $\Rightarrow$ (F) $\Rightarrow$ (D+) $\Rightarrow$ (D) and, if $C$ is weakly compact, (D) $\Rightarrow$ (I).
Evidently, too, the nature of conditions (D+), (D), and (I) is different from
the nature of (FP) and (F): the former are nonseparation assumptions on
the ranges of mappings in $S$, and do not directly refer to fixed points. Never-
theless, our main result is that (I) $\Rightarrow$ (FP) if $X$ is strictly convex and the
mappings in $S$ are nonexpansive. Indeed, $F(S)$ is then a nonexpansive
retract of $C$—the range of a nonexpansive retraction. (For properties of
nonexpansive retracts, see [1], [2], [3].)

\textsuperscript{1} Partially supported by NSF Grant GP-38516.
Theorem 1. If $X$ is strictly convex and $S$ is a compact, convex semigroup of nonexpansive self-mappings of $C$ which satisfies (I), then $F(S)$ is a nonempty nonexpansive retract of $C$.

Proof. Define a partial order $\preceq$ on $S$ by setting $f \prec g$ to mean $\|fx - fy\| \leq \|gx - gy\|$ for all $x, y$ in $C$, with inequality holding for at least one pair $x, y$, and $f \preceq g$ to mean $f < g$ or $f = g$. This order was introduced in [2], [3]. As in the proof of [3, Lemma 2], there exists a minimal element $r$ in $(S, \preceq)$, and each $s$ in $S$ acts as an isometry on $\mathcal{R}(r)$:

$$\|sr(x) - sry\| = \|r(x) - r(y)\|.$$  

(1)

[The proof of Lemma 2 in [3] is inaccurate in that the initial segments $\mathcal{I}_S(g) = \{f \in S: f \preceq g\}$ are not compact, as claimed. However, if $g_1 \prec g_2$ then $\text{cl} \mathcal{I}_S(g_1)$ is compact and is contained in $\mathcal{I}_S(g_2)$, and this is all that is needed to prove the existence of a minimal $r$.]

If $r$ is minimal in $(S, \preceq)$ and $s \in S$, then $\frac{1}{2}s + \frac{1}{2}r \in S$ and

$$\frac{1}{2} \|sr(x) + sry\| - \frac{1}{2} \|r(x) + r(y)\| \leq \|sr(x) - sry\| + \frac{1}{2} \|r(x) - r(y)\|.$$  

(2)

Equality must hold throughout (2) since $r$ is minimal, hence, by (1) and the strict convexity of $X$, $sr(x) - sry = r(x) - r(y)$. Rephrased, if $r$ is minimal in $S$, then each $s$ in $S$ acts as a translation on $\mathcal{R}(r)$.

In particular, $r$ acts as a translation on $\mathcal{R}(r)$. But $\mathcal{R}(r)$ is bounded and $r$-invariant, so this means $r$ acts as the identity on $\mathcal{R}(r)$. Thus $r$ is a (nonexpansive) retraction of $C$ onto $\mathcal{R}(r)$.

Let $r_1, r_2$ be a minimal in $(S, \preceq)$. We claim $\mathcal{R}(r_1) = \mathcal{R}(r_2)$. Indeed, we have already shown that $r_1$ acts as a translation by some vector $v$ on $\mathcal{R}(r_2)$ and as the identity on $\mathcal{R}(r_1)$; but $\mathcal{R}(r_1)$ and $\mathcal{R}(r_2)$ are closed and convex (they are the fixed-point sets of the nonexpansive mappings $r_1$ and $r_2$, and $X$ is strictly convex; see [6]); so condition (1) implies $\mathcal{R}(r_1) \cap \mathcal{R}(r_2) \neq \emptyset$. Thus $v = 0$. That is, $r_1$ acts as the identity on $\mathcal{R}(r_2)$, so that $\mathcal{R}(r_2) \subset \mathcal{R}(r_1)$. By symmetry, $\mathcal{R}(r_1) = \mathcal{R}(r_2)$ as claimed.

Next, we claim that if $r$ is minimal in $(S, \preceq)$ then $\mathcal{R}(r) = F(S)$. Obviously $F(S) \subset \mathcal{R}(r)$. To prove the reverse inclusion, let $s \in S$. By virtue of (1), $sr$ is also minimal in $(S, \preceq)$. But we have shown that minimal elements of $S$ are retractions, all of which have the same range; therefore $sr$ is a retraction of $C$ onto $\mathcal{R}(r)$. If $x \in \mathcal{R}(r)$ then $r(x) = x$ and $sr(x) = x$; so $s(x) = x$. Since this is true for all $s$ in $S$, we have proven $\mathcal{R}(r) \subset F(S)$, and hence $\mathcal{R}(r) = F(S)$.

$F(S)$ is nonempty because, obviously, $\mathcal{R}(r) \neq \emptyset$, and $r$ is a nonexpansive retraction of $C$ onto $F(S)$. Q.E.D.
In practice, the most onerous assumption in Theorem 1 is that $S$ is compact in the topology of weak pointwise convergence. It is usually fairly easy to generate convex semigroups which satisfy (I). For example, suppose $T: C \to C$ is nonexpansive. The existence of a sequence $\{x_n\}$ such that $\lim_n \|x_n - Tx_n\| = 0$ is standard; put $S = \{s: s$ is nonexpansive self-mapping of $C$ and $\lim_n \|x_n - s(x_n)\| = 0\}$. Obviously $S$ is convex and satisfies (D); hence, if $C$ is also weakly compact, $S$ satisfies (I). $S$ is a semigroup because
\[
\|s_1 s_2(x) - x\| \leq \|s_1 s_2(x) - s_1(x)\| + \|s_1(x) - x\| \leq \|s_2(x) - x\| + \|s_1(x) - x\|
\]
whenever $s_1$ is nonexpansive. Evidently $T \in S$, so that a common fixed point of $S$ is a fixed point of $T$. But we are unable to use Theorem 1 to prove the existence of a fixed point of $T$ because apparently $S$ may not be compact.

The situation is different when $C$ is strongly compact.

**Theorem 2.** If $X$ is strictly convex, $C$ is strongly compact, and $S$ is merely a convex semigroup of nonexpansive self-mappings of $C$ which satisfies (I), then $S$ also satisfies (FP).

**Proof.** Since $C$ is compact and $S$ is equicontinuous, the closure $\overline{S}$ of $S$ in $C^C$ is also the closure of $S$ in the topology of uniform convergence, and the weak pointwise convergence topology on $\overline{S}$ is the same as the topology of uniform convergence [5, p. 232]. Obviously $S$ and $\overline{S}$ have the same fixed points. Since mappings in $\overline{S}$ can be uniformly approximated by mappings in $S$, it is easy to see that $\overline{S}$ satisfies (I) if $S$ does. By Theorem 1, therefore, $\overline{S}$ satisfies (FP), hence so does $S$. Q.E.D.

**Example.** (I) does not imply (FP) if $X$ is not strictly convex, even if $C$ is compact. We give an example pattemed after DeMarr [4]. Let $X$ be $R^2$ with the sup norm and let $C$ be the square $\{(x, y): |x| \leq 1, |y| \leq 1\}$. For $0 \leq t \leq 1$ define $f_t(x, y) = (|x| - t, y)$, and put $S = \{f_t: 0 \leq t \leq 1\}$. Since $f_t s = f_t$ and $\lambda f_t + (1 - \lambda) f_s = f_{\lambda t + (1 - \lambda) s}$, $S$ is a convex semigroup. Evidently $S$ is compact and each $f_t$ in $S$ is nonexpansive, (I) is satisfied because the range of $f_t$ is the broken line segment joining $(1 - t, 1)$ to $(- t, 0)$ to $(1 - t, - 1)$, so that $(0, 0) \in \bigcap_{t \in [0, 1]} R(f_t): 0 \leq t \leq 1$. Nevertheless, none of the conditions (FP), (F), (D+), or (D) is satisfied.

**REFERENCES**


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