A NOTE ON A DIFFERENTIAL CONCOMITANT

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ABSTRACT. If \( h \) and \( k \) are vector 1-forms, the vanishing of the concomitant \([h, k]\) is an integrability condition for certain problems on manifolds. In the case that \( h = k \) the vanishing of the Nijenhuis tensor \([h, h]\) implies \( d(tr h) \) is a conservation law for \( h \), provided that \( tr h \) is not constant. When the trace of \( h \) is constant, a conservation law for \( h \) exists if one can find a vector 1-form \( k \) with nonconstant trace such that \([h, k]=0\).

1. Introduction. For any vector 1-form \( h \) there is an associated operator \( d_h \) which is an antiderivation of degree one. In the case that \( h \) is non-singular and has vanishing Nijenhuis tensor \([h, h]\), the operator \( d_h \) satisfies a Poincaré lemma and thus provides an \( h \)-dependent version of de Rham's theorem. Such an operator is an example of a special type of derivation whose basic properties were studied in a definitive paper by A. Frölicher and A. Nijenhuis [3]. When \( h \) is the identity on 1-forms, the operator \( d_h \) reduces to the usual exterior differentiation operator \( d \). This derivation \( d_h \) and other operators related to it may be used to give an alternative description of the differential concomitant \([h, k]\) of vector 1-forms \( h \) and \( k \). The vanishing of this concomitant yields the main result of this paper. It is an identity which simply says that the composition \((d \circ \text{trace})\) acts as a first order derivation on the composition \(hk\).

The derivational property of \( d \circ \text{trace} \) is applicable in the study of conservation laws. With it one can establish the existence of conservation laws in a constructive manner which has the added advantage of giving a global result. Since conservation laws have usually been mathematically studied in only a local manner, global results are few. For example, in general relativity the effect of a global conservation law (Einstein's equations) on the manifold topology is of current physical interest (see [2]). Implicit in any global conservation law problem is an interaction with topology; the derivational identity yields a possible method which can be applied in such global problems.

2. Preliminaries. Let \( A \) denote the algebra of \( C^\infty \) functions on a compact, orientable, \( n \)-dimensional Riemannian manifold \( M \) without boundary.

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Let $E$ denote the $A$-module of differential 1-forms on $M$. A vector 1-form $h \in \text{End}_A E$ induces endomorphisms $h^{(q)} \in \text{End}_A (\wedge^q E)$ for any nonnegative integer $q$, and the $h^{(q)}$ are defined by setting $h^{(q)} = 0$ if $q > p$ and

$$h^{(q)} (\phi^1 \wedge \cdots \wedge \phi^p) = \frac{1}{(p-q)!q!} \sum |\pi| h \phi^{(1)} \wedge \cdots \wedge h \phi^{(q)} \wedge \phi^{(q+1)} \wedge \cdots \wedge \phi^{(p)}$$

if $0 \leq q \leq p$, where $\phi^i \in E$, $\pi$ runs through all permutations of $(1, \ldots, p)$ and $|\pi|$ denotes the sign of the permutation. The transformation $h^{(0)}$ is taken to be the identity mapping on $\wedge^p E$.

In the case where $q = p \leq n$, the operator $h^{(p)}$ is locally represented by an $\binom{n}{p} \times \binom{n}{p}$ matrix $[h^{(p)}]$ relative to some local basis of $p$-forms. If $[h]$ denotes an $n \times n$ matrix which locally represents $h$, then it can be shown that

$$\det [h^{(p)}] = (\det [h])^{\binom{n-p}{p}}$$

and hence $h$ is invertible on 1-forms if and only if $h^{(p)}$ is invertible on $p$-forms.

An alternating derivation $d_h: \wedge E \to \wedge E$ is obtained from $h$ and an exterior derivation $d$ by setting $d_h = h^{(1)}d - dh^{(1)}$. Thus when $h$ is the identity, $d_h$ reduces to $d$. The Nijenhuis tensor can be extended as a derivation on $\wedge E$. On $p$-forms one thus obtains the formula

$$[h, h] = h^{(2)}d + d_h h^{(1)} + dh^{(2)},$$

which can be rewritten in the form

$$[h, h] = \frac{1}{2} [d, h] - (d_h)_h.$$ 

In a similar way the concomitant $[h, k]$ of vector 1-forms defined by the equation

$$[h, k] = \frac{1}{2} \{ [h^{(1)}, k^{(1)}] - (hk)^{(1)} \} d$$

$$+ [h^{(1)}dk^{(1)} + k^{(1)}dh^{(1)}] - [d(hk)^{(1)} + (k^{(1)}h^{(1)})]$$

can be rewritten in the form

$$[h, k] = \frac{1}{2} [d, h k] - (d_h)_k.$$ 

3. The main identity. Let $\omega^1, \ldots, \omega^n$ be a local orthonormal basis of differential 1-forms and suppose that $\omega^1 \wedge \cdots \wedge \omega^n$ agrees with an orientation of $M$. Suppose also that the vector 1-form $h$ is locally specified by setting $h \omega^i = h^i \omega^i$, where $h^i \in A$ and $1 \leq i, j \leq n$, and the Einstein summation convention has been invoked. Since the orientation is specified, the Hodge star operator $*$: $\wedge^p E \to \wedge^{n-p} E$ is determined by setting

$$*(\omega^i_1 \wedge \cdots \wedge \omega^i_p) = \epsilon_{i_1 \cdots i_p} \omega^1_{i_1} \wedge \cdots \wedge \omega^{i_{n-p}}_p.$$
where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_{n-p}$ are complementary sets of positive integers and $\epsilon_{i_1 \ldots i_p}$ is +1 or -1 if the permutation $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ of the integers $(1, 2, \ldots, n)$ is even or odd respectively. In the sequel the trace of $h$ and the transpose of $h$ will be denoted by $\text{tr} \ h$ and $h^t$ respectively. The following lemma is a key to establishing our identity. The proof of the lemma appears in [1].

**Lemma 3.1.** For any vector 1-form $h$,

$$h^{(1)*} + *h^{(1)*} = (\text{tr} \ h)*.$$

Since the manifold $M$ is assumed to be compact, an inner product on $\wedge^p E$ is defined by setting

$$(a, \beta) = \int_M a \wedge *\beta$$

for any $p$-forms $a$ and $\beta$. With respect to this inner product the adjoint of $h^{(p)}$ is $h^{(p)*}$ for $p = 0, 1, \ldots, n$.

The codifferential $\delta$ is defined to be the adjoint of the exterior derivative $d$. The adjoint of $d_h = h^{(1)*}d - dh^{(1)}$ is then easily seen to be $\delta_h = \delta h^{(1)*} - h^{(1)}\delta$. The operator $\delta_h$ is called the $h$-codifferential. As a consequence of Stokes' theorem one obtains the relation $\delta = (-1)^{np+n+1} \ast d \ast$ on $p$-forms. The corresponding expression for $\delta_h$ is obtained in the following lemma.

**Lemma 3.2.** If $h$ is a vector 1-form, then on $p$-forms

$$\delta_h = (-1)^{np+n+1}\ast d \ast h + d(\text{tr} \ h) \wedge \ast.$$

**Proof.** On $p$-forms one obtains

$$\delta_h = \delta h^{(1)*} - h^{(1)}\delta = (-1)^{np+n+1}\ast d \ast h^{(1)*} - h^{(1)*}d \ast,$$

and thus Lemma 3.1 may then be applied to yield

$$\delta_h = (-1)^{np+n+1}\ast d \ast [\text{tr} \ h* - h^{(1)*}] - [\ast \text{tr} \ h - \ast h^{(1)*}]d \ast$$

$$= (-1)^{np+n+1}\ast d \ast [\text{tr} \ h* - \ast h^{(1)*}] - \ast h^{(1)*}d \ast$$

$$= (-1)^{np+n+1}\ast d \ast [\text{tr} \ h + d(\text{tr} \ h) \wedge \ast].$$

The next lemma establishes a formula for $\text{adj}[h, k]$ for any $h, k \in \text{End}_A E$; this formula is analogous to the formula for $\delta_h$ in the preceding lemma.

**Lemma 3.3.** For any $h, k \in \text{End}_A E$,

$$\text{adj}[h, k] = (-1)^{np+1}\ast [h, k] + \frac{1}{2}\{d(\text{tr} \ hk) - hd(\text{tr} \ k) - kd(\text{tr} \ h)\} \wedge \ast$$

on $p$-forms.

**Proof.** Since
\[ *[h, k] = \frac{1}{2} [d_{hk} - (d^*_h)_k] = \frac{(-1)^p (n-p)}{2} *[d_{hk}^* - (d^*_h)_k]^* \]

on \(p\)-forms, it suffices to compute \(*d_{hk}^* \text{ and } *(d^*_h)_k^* \text{ on } (n-p)\) 1-forms. Thus, as a consequence of Lemma 3.2, one has

\[ *d_{hk}^* = (-1)^p + 1 \text{det} \] and

\[ *(d^*_h)_k^* = *[k(1)d_h - d^*_h k(1)]^* \]

\[ = \{\text{tr } k - k(1)\}^* d_{hk}^* - *d_{hk}^* \{\text{tr } k - k(1)\}^* \]

by Lemma 3.1. The expression for \(*d_{hk}^* \text{ then reduces to} \]

\[ *(d^*_h)_k^* = (-1)^p + 1 \text{det} \]

\[ + * \left\{ k^d(\text{tr } h) + \text{hd}(\text{tr } k) \right\}^* \]

and, hence,

\[ *[h, k] = (-1)^p + 1 \text{det} \]

\[ + * \left\{ k^d(\text{tr } h) + \text{hd}(\text{tr } k) - d(\text{tr } hk) \right\}^* \]

Thus

\[ *[h, k] = (-1)^p + 1 \text{det} \]

\[ + * \left\{ k^d(\text{tr } h) + \text{hd}(\text{tr } k) - d(\text{tr } hk) \right\}^* \]

and the desired result is then obtained by an application of the Hodge star operator *, which is an isomorphism, on the right side of the above expression.

If \([h, k]\) vanishes, then Lemma 3.3 yields an identity which is the main result of this paper. The identity is given in the following proposition.

**Proposition 3.4.** If \([h, k] = 0\), then \(d(\text{tr } hk) = h(\text{tr } k) + k(\text{tr } h)\). Thus \(D = d \circ \text{trace} : \text{End}_A E \rightarrow E\) is a first order derivation on compositions of vector 1-forms \(h \text{ and } k\) whose differential concomitant \([h, k]\) vanishes.

One can then study equations of the form \(Dh = \alpha\) for some \(\alpha \in E\) where a solution \(h \in \text{End}_A E\) is desired. This equation is analogous to first order differential equations on the real line. The study of such equations will not be pursued here. However, the fact that \(D\) acts analogously to first derivatives on polynomial functions of one real variable will be shown in the following corollaries to Proposition 3.4.

**Corollary 3.5.** If \([h, h] = 0\), then for any nonnegative integers \(i\) and \(j\),

\[ h^i d(\text{tr } h^j) + h^j d(\text{tr } h^i) = d(\text{tr } h^{i+j}) \]

Proof. The vanishing of \([h, h]\) implies the vanishing of the concomitant \([h^i, h^j]\) for any nonnegative integers \(i\) and \(j\). Thus, as a consequence of Proposition 3.4 with \(h^i\) and \(h^j\) in place of \(h\) and \(k\), the corollary is established.
Corollary 3.6. If \([h, h] = 0\), then for any nonnegative integer \(i\),
\[ h^i d(\text{tr} \ h) = (i + 1)^{-1} d(\text{tr} \ h^{i+1}). \]

Proof. If \(i = 1\), then \(h d(\text{tr} \ h) = \frac{1}{2} d(\text{tr} \ h^2)\) as a consequence of Corollary 3.5. If \(h^j d(\text{tr} \ h) = (j + 1)^{-1} d(\text{tr} \ h^{j+1})\) for some positive integer \(j > 1\), then
\[ h^{j+1} d(\text{tr} \ h) = \frac{1}{j+1} h d(\text{tr} \ h^{j+1}) = \frac{1}{j+1} \{ -h^{j+1} d(\text{tr} \ h) + d(\text{tr} \ h^{j+2}) \} \]
as another consequence of Corollary 3.5. Hence
\[ h^{j+1} d(\text{tr} \ h) = (j + 2)^{-1} d(\text{tr} \ h^{j+2}). \]
This induction argument establishes the corollary.

Corollary 3.7. If \([h, h] = 0\), and if \(h\) has constant trace, then the coefficients in the characteristic polynomial of \(h\) are constant.

Corollary 3.7 is an immediate consequence of the fact that if \(h\) has constant trace, then all the positive powers of \(h\) also have constant trace.

In the case that the coefficients of the characteristic polynomial of a vector 1-form \(h\) are all constant, a result analogous to the formula \(\delta^* = (-1)^{p+1} \pi^* d\) on \(p\)-forms is obtained. Thus we have

Corollary 3.8. If the coefficients of the characteristic polynomial of \(h\) are constant, then
\[ *[h, h] = (-1)^{p+1} (\text{adj} [h, h])*. \]

Proof. Let \(h^n = a_0 I + a_1 h + \ldots + a_{n-1} h^{n-1}\) be the characteristic equation of \(h\). Since \(a_{n-2} = \frac{1}{2} \text{tr} (h^2) - a_{n-1}^2\), and \(\text{tr} h = a_{n-1}\), it follows that \(\text{tr} (h^2)\) is constant and, hence, \(h d (\text{tr} h) = \frac{1}{2} d (\text{tr} h^2) = 0\). Thus \(*[h, h] = (-1)^{p+1} \text{adj} [h, h]*\) on \(p\)-forms as a consequence of Lemma 3.3.

This section is concluded with an alternate version of the identity in Corollary 3.6. Since the traces of the various powers of \(h\) are expressible in terms of the coefficients \(a_i\), the following proposition may be obtained.

Proposition 3.9. If \([h, h] = 0\) and \(h^n = a_0 I + a_1 h + \ldots + a_{n-1} h^{n-1}\) is the characteristic equation of \(h\), then
\[ h d a_i = d a_{i-1} + a_i d a_n, \quad i = 0, 1, \ldots, n-1 \quad (da_{n-1} = 0). \]

Proof. For the case that \(h\) is cyclic, a proof appears in [6]. Otherwise the identity in Corollary 3.6 may be used. Thus \(h d (\text{tr} h) = \frac{1}{2} d (\text{tr} h^2)\) and, since \(\text{tr} h = a_{n-1}\) and \(\text{tr} h^2 = 2a_{n-2} + a_{n-1}^2\), one obtains \(h d a_{n-1} = d a_{n-2} + a_{n-1}^2 d a_{n-1}\). Hence, the case \(i = n - 1\) in the proposition is established. If \(h\) is applied to the above result, the relation obtained is
\[ h d a_{n-2} = h^2 d a_{n-1} - a_{n-1} h d a_{n-1} \]
\[ = h^2 d (\text{tr} h) - a_{n-1} h d (\text{tr} h) = (1/3) d (\text{tr} h^3) - (a_{n-1}/2) d (\text{tr} h^2). \]
since $\text{tr } h^3 = 3a_{n-3} + 3a_{n-1}a_{n-2} + a_{n-1}^3$, the case $i = n - 2$ is established. Thus

$$h da_{n-2} = da_{n-3} + a_{n-2}da_{n-1}.$$ 

The remaining identities can be established by repeated use of Corollary 3.6, though the calculation is tedious because of the form of the expressions for $\text{tr } h^i$.

**Corollary 3.10.** If the Nijenhuis tensor $[h, h]$ vanishes identically, then the coefficients $a_i$ in the characteristic equation for $h$ are functionally independent if and only if $h$ is cyclic with generator $d(\text{tr } h)$.

**Proof.** It is sufficient to establish the relation

$$d(\text{tr } h) \wedge h d(\text{tr } h) \wedge \cdot \cdot \cdot \wedge h^{n-1}d(\text{tr } h) = da_{n-1} \wedge da_{n-2} \wedge \cdot \cdot \cdot \wedge da_0,$$

which is a consequence of the proposition.

4. Conservation laws. The conservation laws of physics can be expressed in conservation law form as a differential equation [4], and also as a relation between vector 1-forms and differential forms [5]. The use of vector 1-forms and differential forms trivially extends the problem to a global one on manifolds. Global conservation laws are of interest in physics as is evident from [2], but to date do not appear to have been studied in a general mathematical context. With the identity stated in Corollary 3.4 the global conservation law problem can be studied.

On a differentiable manifold a conservation law for a vector 1-form $h$ is defined as any exact differential form $\theta$ such that $h \theta$ is also exact. Thus if $\theta = df$ for some differentiable function $f$, then $\theta$ is a conservation law for $h$ if there is a differentiable function $g$ such that $h df = dg$. The analysis may be divided into two cases insofar as the existence of conservation laws is concerned. The cases correspond to constant and nonconstant trace for $h$.

If $h$ has trace which is not constant, then Corollary 3.6 says that $d(\text{tr } h)$ is a conservation law for any positive power of $h$. Moreover if $h$ is cyclic with $d(\text{tr } h)$ as a generator, then Corollary 3.1 implies that a basis of conservation laws can be obtained. This discussion is summarized in the following statement.

**Proposition 4.1.** Let $M$ be a compact orientable $n$-dimensional Riemannian manifold without boundary and let $h$ be a vector 1-form whose trace is not constant on $M$. If the concomitant $[h, h]$ vanishes on $M$, then $d(\text{tr } h^i)$ with $i = 1, 2, \ldots$ is a conservation law for $h$, and for fixed $i$, $d(\text{tr } h^i)$ is a conservation law for $h^i$, with $j = 1, 2, \ldots$. Finally if $h$ is cyclic with generator $d(\text{tr } h)$, then the collection $d(\text{tr } h^i), i = 1, 2, \ldots, n$, is a basis of conservation laws for $h$.

The analysis for the case in which $\text{tr } h$ is constant is not as simple.
The above reasoning can be extended provided that a nontrivial solution can be obtained for a certain system of differential equations. The solution of global differential equations is, of course, limited by topological considerations. In particular, it is possible to find another vector 1-form \( k \) with nonconstant trace such that the concomitant \([h, k]\) vanishes, then the identity of Proposition 3.4 yields the statement that \( d(\text{tr } h) \) is a conservation law for \( h \). In local coordinates the vanishing of \([h, k]\) leads to a system of \( n^2 \) equations (corresponding to \( n \) basis elements and \( \binom{n}{2} \) elements in \( E \wedge E \)) in \( n^2 \) unknowns. This system may be written out explicitly if we assume that \((dx^1, \ldots, dx^n)\) is a local basis of 1-forms, that \( hdx^i = h^i_\alpha dx^\alpha \), \( kdx^i = k^i_\alpha dx^\alpha \), and that \( \alpha(j) = \partial(\alpha)/\partial x^i \). Thus the condition \([h, k] = 0\) leads to a local system of the form

\[
2(h^i_\beta \delta^\beta_\alpha - h^i_\alpha \delta^\beta_\beta)k^i_\beta = \left(2h^i_\beta \delta^\beta_\alpha + h^i_\alpha \delta^\beta_\beta - 2h^i_\beta \delta^\beta_\beta - h^i_\beta \delta^\alpha_\alpha - h^i_\beta \delta^\alpha_\alpha \right)k^i_\beta
\]

for \( 1 \leq j < l < n \) and \( 1 \leq i \leq n \). For \( n \geq 3 \) the system \([h, k] = 0\) is generally overdetermined but may still have nontrivial solutions. To summarize this section one has

**Proposition 4.2.** Let \( M \) be a compact orientable \( n \)-dimensional Riemannian manifold without boundary and let \( h \) be a vector 1-form on \( M \). If there is a vector 1-form \( k \) with nonconstant trace and vanishing concomitant \([h, k]\), then \( d(\text{tr } k) \) is a conservation law for \( h \), provided \( \text{tr } h \) is constant.

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