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THE INTERSECTION OF THE SUBGROUPS OF FINITE INDEX IN SOME FINITELY PRESENTED GROUPS

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ABSTRACT. We consider the intersection of the subgroups of finite index in some finitely presented non-Hopfian groups.

The effort to establish the existence of a finitely generated non-Hopfian group seems to date back to the 1940's. R. Baer [5] was one of the first to consider this problem. Since then, several examples of finitely presented non-Hopfian groups have been given. If G is any finitely generated group and if α is a surjective endomorphism of G, α induces in a natural way a surjective endomorphism $\bar{\alpha}$ of G/N, where N is the intersection of the subgroups of finite index of G. But G/N is residually finite and a finitely generated residually finite group is Hopfian. Hence $\bar{\alpha}$ is an automorphism and, consequently, the kernel of α is in N. Similarly if K_i is the kernel of α^i, $K_i \subset N$ so that if $K = \bigcup_{i \geq 1} K_i$, then $K \subset N$.

Professor W. Magnus has raised the question concerning under what circumstances $K = N$. The object of this note is to examine some of the known finitely presented non-Hopfian groups G and to show that for these groups there exists an α such that K is as large as possible, namely $K = N$.

Theorem. *Consider the following groups G with the respective endomorphisms α:*

(I) $\langle a, b;\ a^{-1}b^l a = b^m\rangle$, *$l$, m relatively prime,* $a\alpha = a$, $b\alpha = b^l$;

(II) $\langle a, b, d;\ [a, b, b] = [a, b, a] = [a, d] = 1,\ a = [a^2, b, d]\rangle$, $a\alpha = a^2$, $b\alpha = b$, $d\alpha = d$;

(III) $\langle a, b, c;\ a^{-1}ca = b^{-1}cb = c^r\rangle$, $r \geq 2$, $a\alpha = a$, $b\alpha = b$, $c\alpha = c^r$.

Then in each case above, $K = N$. In (I), $K = N = G''$.

Proof. For the proofs that α gives a surjective endomorphism which is not an automorphism, see [9], [7] and [18] respectively. To show $K = N$, it suffices to show $N \subset K$. Let $g \in N$. We consider (I). If B is the normal subgroup generated by b, G/B is residually finite so that $g \in B$. Hence we can express g as a word in the elements $x_i = a^i b a^{-i}$, $g = w(x_i)$. Let r be the largest positive integer such that x_r appears in w. Let $-s$ be the smallest negative integer such that x_{-s} appears in w. Let θ be the inner automor-

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phism: $u \to a^{-r}ua^r$. Then by using the relation $s_1^l = x_{i+1}^m$, one can easily check that $g\alpha^{r+s}\theta$ is a power of b. But $N\alpha = N$, $N\theta = N$ so $g\alpha^{r+s}\theta$ is a power of b in N. However, in any finitely generated group G, $N \subset G''$ since G/G'' is a residually finite group. Hence $g\alpha^{r+s}\theta = b^j \in G'' \subset B'$. However, B/B' is isomorphic to the additive group of rationals generated by $(l/m)^i$, $i = 0, 1, 2, \ldots$, under the map $x_iB' \to (l/m)^i$ [4, p. 477] so that $b^j \in B'$ implies $j = 0$. Hence $g\alpha^{r+s}\theta = 1$ and, hence, $g\alpha^{r+s} = 1$, so that $K = N$. To show that $N = G''$ it suffices to show that G'' is contained in any normal subgroup L of finite index in G. But note that $a^{-k}b^{l^k}a^k = b^{m^k}$ so that in any finite quotient group $\bar{G}$ of G, the order j of b is a divisor of $m^k - l^k$ for some k so that $(j, l) = 1$. Hence in this quotient group we may write $b = b^{sl}$ so that $a^{-1}ba = b^{ms}$. This implies that $\bar{G}$ has a cyclic commutator subgroup so that if $G/L = \bar{G}$, then $G'' \subset L$.

For the group (II) we define $a_0 = a$ and $a_{n+1} = (a_n, b, d)$. Now we assert the modulo K, generators for A, the normal subgroup generated by a, are $S = \{a_n, (a_n, b) = c_n, n \geq 0\}$. For one set of generators of A consists of the elements $w^{-1}aw$, where $w = w(b, d)$ is a freely reduced word in b and d. We show by induction on the length of the word w that $w^{-1}aw$ may be expressed as a product of elements in S and elements in K. If the length of w is zero, so that w is the empty word, our assertion is apparent.

Assume the assertion is true for words of length n. Let w be a word of length $n + 1$. Write $w = ux$ where u is of length n and x is either b, d, b^{-1} or d^{-1}. Then by our inductive assumption,

$$u^{-1}au = s_1^{e_1}s_2^{e_2} \cdots s_j^{e_j}k, \qquad e_i = \pm 1,$$

for some $s_i \in S$ and $k \in K$. Hence $w^{-1}aw = (x^{-1}s_1x) \cdots (x^{-1}s_jx)(x^{-1}kx)$. Now since K is a normal subgroup, $x^{-1}kx \in K$. Hence, we need only show that if $x \in S$, $x^{-1}sx$ may be expressed as a product of elements s_*, s_*^{-1}, k_* where $s_* \in S$, $k_* \in K$.

We first consider $s = a_n$. Note $a_n\alpha^n = a$. Hence, $(d, a_n)\alpha^n = (d, a) = 1$, so that $d^{-1}a_nd = a_n \bmod K$ and $da_nd^{-1} = a_n \bmod K$. Also $ba_nb^{-1} = (b^{-1}, a_n^{-1})a_n$. However, $[(b^{-1}, a_n^{-1})c_n]\alpha^{n-1} = 1$, so that $(b^{-1}, a_n^{-1}) = c_n^{-1} \bmod K$, so that $ba_nb^{-1} = c_n^{-1}a_n \bmod K$. Similarly, $b^{-1}a_nb = c_na_n \bmod K$. Now to consider $s = c_n$, note

$$d^{-1}c_nd = (d, c_n^{-1})c_n.$$

But we claim

$$(d, c_n^{-1}) = (d, c_n)^{-1} \bmod K.$$

To see this, note that if $\bar{a} = b^2d^{-1}(a, b)db^{-2}$, then a brief calculation shows $\bar{a}^2 = a$ [7, p. 197] so that $\bar{a}\alpha = \bar{a}^2 = a$.

Now, note that $c_0^{-1}dc_0 = d\bar{a}$ mod K and $(d, \bar{a}) = 1$ mod K. This implies that $(d, c_0)^{-1} = (d, c_0^{-1})$ mod K. Now if we apply α^n to (d, c_n) and (d, c_n^{-1}), respectively, we may deduce (2), so that from (1) and (2) we see $d^{-1}c_n d = a_{n+1}c_n$ mod K. Similarly $dc_n d^{-1} = a_{n+1}^{-1}c_n$ mod K. Clearly, $b^{-1}c_n b = c_n$ mod K so that our assertion about S is valid.

Now note that G/A is a free group of rank two which is consequently residually finite. This implies that $N \subset A$. Hence if $w \in N$, we may write $w = s_1^{\epsilon_1}s_2^{\epsilon_2} \cdots s_j^{\epsilon_j}k$, where $s_i \in S$ and $k \in K$. Note that $a_n\alpha^n = a$, $c_n\alpha^n = (a, b)$, $n \geq 1$, so that we may find a positive integer r such that $w\alpha^r = a^j(a, b)^{-m} \in N$. Hence in every finite quotient $\bar{G}$ of G we have $a^j = (a, b)^m$. Hence in $\bar{G}$, $(a^j, b) = (a, b)^j = 1$. Hence in $\bar{G}$, $a^j = (c_0^{2j}, d) = 1$ and $a^m = (c_0^{2m}, d) = 1$. Hence in G, $a^j \in N$ and $a^m \in N$. We will show this implies $j = m = 0$ so that $w\alpha^r = 0$. To do this, for odd n let L_n be the normal subgroup of G generated by a^{n^2}, b^{n^2}, d^{n^2}, $n > 1$. Let

$$P_n = \langle a, b; (a, b, a) = (a, b, b) = a^{n^2} = b^{n^2} = 1\rangle,$$

$$M_n = \langle c, d; (c, d, c) = (c, d, d) = c^{n^2} = d^{n^2} = 1\rangle.$$

Let H_n be the subgroup of P_n generated by (a^2, b) and a. Let J_n be the subgroup of M_n generated by (c, d) and c.

Form the free product of P_n and M_n amalgamating H_n and J_n as $c = (a^2, b)$ and $a = (c, d)$. This free product with $H_n = J_n$ is just G/L_n. Now a free product of finite groups with a subgroup amalgamated is residually finite. If $n > 1$ is odd, $a^n \neq 1$ in G/L_n so that we can find a finite quotient of G/L_n (and hence of G) in which $a^n \neq 1$. Hence, if t is odd, $a^t \notin N$ in G. Moreover, if t is even, $t \neq 0$, $t = 2^q f$, f odd, then $a^t \notin N$, for as can be checked from the defining relations of G, the order of a in every finite quotient of G is odd so that $a^t \in N$ would imply $a^f \in N$.

For the group (III), if C is the normal subgroup generated by c, we see that G/C is a free group of rank two so that $N \subset C$. Moreover, generators for C are $w^{-1}cw$, where $w = w(a, b)$ is a freely reduced word in a and b. Now we claim that modulo K generators of C are the elements $c_j = a^j c a^{-j}$, $j \geq 0$. To show this we show by induction on the length of the word w that any element $w^{-1}cw$ is expressible in terms of the c_j and elements of K. If the length of w is zero, that is, if w is the empty word, this is clear. To complete the induction it suffices, as in the discussion for the group (II), to show that $x^{-1}c_j x$ is expressible in terms of the c_t and elements in K, where x is one of the elements a, b, a^{-1}, b^{-1}. To this end note that $ac_j a^{-1} = c_{j+1}$ and $a^{-1}c_j a = c_{j-1}$ if $j \geq 1$ while $a^{-1}c_0 a = c_0^r$. Also we note that if $b_j = b^j c b^{-1}$, $j \geq 0$, then $b_j = a_j$ mod K. For note, $b_j\alpha = b^j c^r b^{-j} = b^j b^{-1} c b b^{-j} = b_{j-1}$, $j \geq 1$. Hence $b_j\alpha^j = b_0 = c$. Similarly $c_j\alpha^j = c$ so that $(b_j c_j^{-1})\alpha^j =$

1. Hence modulo K we may write $bc_jb^{-1} = bb_jb^{-1} = b_{j+1} = c_{j+1}$ and $b^{-1}c_jb = b^{-1}b_jb$. The latter is $b_{j-1} = c_{j-1}$ if $j \geq 1$ and is $b_0^r = c_0^r$ if $j = 0$. This completes the proof that elements of N are expressible in terms of the c_j and elements in K.

In view of the fact that $c_j\alpha^j = c$, it follows that if $w \in N$, then $w\alpha^t = c^u$ for some positive integer t. Hence $c^u \in N$. However, if we consider the groups

$$P_n = \langle a, c;\ a^{-1}ca = c^r,\ a^n = 1\rangle, \qquad n \geq 2,$$

$$M_n = \langle b, d;\ b^{-1}db = d^r,\ b^n = 1\rangle, \qquad n \geq 2,$$

then c and d have the same orders and $c^n \neq 1$. Since P_n and M_n are finite groups, their free product with the amalgamation $c = d$ is residually finite. However this free product yields a homomorphic image of G. Hence if $c^u \in N$ in G, we must have $u = 0$.

Problem. Does every finitely presented non-Hopfian group have a surjective endomorphism α for which $K = N$?

Some of the references listed below are not referred to in the present paper but are relevant to the topic of Hopficity. References for any assertions made about residually finite groups can be found in the survey paper [30].

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