IDENTITIES IN COMBINATORICS. II:
A \( q \)-ANALOG OF THE LAGRANGE INVERSION THEOREM

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ABSTRACT. A \( q \)-analog of Lagrange's inversion theorem is obtained. It is applied to give a new proof of an expansion theorem due to Carlitz and to obtain formulae for certain combinatorial numbers studied by Carlitz.

1. Introduction. In [3], L. Carlitz addressed the question of possible \( q \)-analogs for the celebrated Lagrange inversion formula [1, p. 158]:

\[
(1.1) \quad f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!(\phi(x))^n} \left[ \frac{d^n}{dt^n} \left( f'(t)(\phi(t))^n \right) \right]_{t=0},
\]

a formula valid provided \( f(x) \) and \( \phi(x) \) are analytic around \( x = 0 \) and \( \phi(0) \neq 0 \).

Carlitz [3, p. 206, equation (1.11)] was able to show that

\[
(1.2) \quad f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{(q)_n (x)_n} [\Delta^{n-1} (\Delta f(t) \cdot (t)_n) ]_{t=0},
\]

where \( (x)_n = (1-x)(1-xq) \cdots (1-xq^{n-1}) \) and \( \Delta f(t) = t^{-1}(f(t) - f(tq)) \). This result is a \( q \)-analog of (1.1) in the case \( \phi(x) = 1 - x \).

In [4], Carlitz asks for a general \( q \)-analog of Lagrange inversion. Namely he asks for a formula for the \( C_n \) in

\[
(1.3) \quad F(x) = \sum_{n \geq 0} \frac{C_n x^n}{(q)_n \phi(xq) \phi(xq^2) \cdots \phi(xq^n)}, \quad C_0 = F(0).
\]

In particular he asks for an operational formula for computing the \( C(n|q) \) in

\[
(1.3) \quad x = \sum_{n=1}^{\infty} x^n(xq)_n C(n-1|q),
\]

and he points out that

\[
C(n-1|q) \neq \frac{(1-q)}{(q)_n} [\Delta^{n-1} (xq)^{-1}] = q^{n-1} \frac{(1-q)}{(1-q^n)} \left( \frac{2n-2}{n-1/q} \right),
\]

where \( \binom{n}{m}_q = (q)_n (q)_m^{-1} \binom{n}{m}_{n-m} \).

In [2], Carlitz discusses \( C(n|q) \) extensively, which is initially defined by

\[
C(n|q) = \sum_{(a_i)} q^{a_1 + a_2 + \cdots + a_n}
\]

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summed over all \((a_1, a_2, \ldots, a_n)\) with \(1 \leq a_1 \leq a_2 \leq \cdots \leq a_n\) and \(a_i \leq i\) for each \(i\).

When \(q = 1\), then, of course, the Lagrange inversion formula (1.1) is applicable to (1.3) and indeed \(C(n) = (n + 1)^{-1}(2n)_n\), the \(n\)th Catalan number.

In this paper we shall (in §2) prove the following \(q\)-analog of the Lagrange inversion formula (1.1); our result reduces to (1.2) in the case \(\phi(x) = 1 - xq^{-1}\) and it reduces to (1.1) when \(q \to 1\). Before stating our main result we require the following notation:

\[
\Delta_{(n)} f(t) = \frac{f(t) - f(tq^n)}{t}.
\]

Note that \(\Delta = \Delta_{(1)}\); indeed \(\Delta_{(n)}\) is just \(q\)-differentiation to the base \(q^n\). Furthermore \(\Delta_{(0)} f(x) = 0\).

**Theorem 1.** Let \(F(x)\) and \(\phi(x)\) be analytic around \(x = 0\) with \(\phi(0) \neq 0\), and let \(k\) be a nonnegative integer. Define

\[
\delta_r = (q)^{r-1} \cdot \{\text{Coefficient of } t^{r-1} \text{ in } (\Delta F(t)) \cdot \phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})\}
\]

and for \(j \leq r\),

\[
\epsilon_j(r) = -(q)^{r-j} q^{r-j} \cdot \left\{\text{Coefficient of } t^{j-2} \text{ in } (\Delta_{(r-j)} f(tq^k)) \cdot \frac{\phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})}{\phi(tq)\phi(tq^2) \cdots \phi(tq^{r-1})} \right\}_{t=0}.
\]

Then, for \(|q| < 1\),

\[
(1.4) \quad F(x) = F(0) + \sum_{n=1}^{\infty} \frac{x^n C_n}{(q)_n \phi(xq)\phi(xq^2) \cdots \phi(xq^n)},
\]

where

\[
C_n = \begin{bmatrix}
1 & 0 & 0 & 0 & \delta_0 \\
\epsilon_1(1) & 1 & 0 & 0 & \delta_1 \\
\epsilon_2(2) & \epsilon_1(2) & 1 & 0 & \delta_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\epsilon_n(n) & \epsilon_{n-1}(n) & \epsilon_{n-2}(n) & \cdots & \epsilon_1(n) & \delta_n
\end{bmatrix}.
\]

When \(q \to 1\), we shall show that \(\epsilon_j(n)(q)^{-1} \to 0\) whenever \(j < n\); hence it follows that
\[ C_n(q)_n^{-1} \to n!^{-1} [(d^{n-1}/dt^{n-1})(f(t)(\phi(t))^n)]_{t=0} \]
as \( q \to 1 \) (see (2.4) at the end of §2). Thus Theorem 1 is the \( q \)-analog of Lagrange inversion.

In §3, we shall apply Theorem 1 to prove (1.2) and certain formulae related to Carlitz's work in [3]. In §4, we shall apply Theorem 1 to the \( C(n|q) \).

2. Proof of Theorem 1. The elementary proof of (1.1) found in [1, §55.1] will be adapted to our needs here. We begin with the observation that the \( C_n \) in (1.4) must exist since \( |x^n/\phi(xq) \cdots \phi(xq^n)| < \rho^n < 1 \) for \( x \) in some sufficiently small neighborhood of 0. Replacing \( x \) by \( t \) in (1.4) and applying \( \Lambda \) we see that

\[
\Delta r(t) = \sum_{n \geq 0} \frac{C_n t^{n-1}}{(q)_n} \left\{ \frac{1}{\phi(tq) \phi(tq^2) \cdots \phi(tq^n)} - \frac{q^n}{\phi(tq^2) \phi(tq^3) \cdots \phi(tq^{n+1})} \right\}
\]

\[
= \sum_{n \geq 0} \frac{C_n t^{n+1} q^n [t^{-1} - n \phi(tq^{n+1})]}{(q)_n \phi(tq) \phi(tq^2) \cdots \phi(tq^{n+1})}
\]

Hence

\[
(\Delta F(t)) \phi(tq^k) \phi(tq^{k+1}) \cdots \phi(tq^{k+r-1}) t^{-r}(q)_{r-1}
\]

\[
= -(q)_{r-1} \sum_{n \geq 0} \frac{C_n t^{n-r+1} q^n (\Delta_{(n)} t^{-1} \phi(tq^n)) \phi(tq^k) \phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})}{(q)_n \phi(tq) \phi(tq^2) \cdots \phi(tq^{n+1})}
\]

We now ask for the coefficient of \( t^{-1} \) on each side of the above identity. On the left-hand side we see that \( \delta_r \) is the coefficient of \( t^{-1} \). On the right-hand side we see that for \( n > r \) each term is analytic in \( t \) and so the coefficient of \( t^{-1} \) is zero; hence we need only examine those terms with \( n \leq r \). When \( n = r \), the term is

\[
\frac{C_r t^{-1} \phi(tq^{r+1}) - q^r \phi(tq) \phi(tq^k) \cdots \phi(tq^{k+r-1})}{(1 - q^r) \phi(tq) \phi(tq^2) \cdots \phi(tq^{r+1})} = \frac{C_r t^{-1}}{1 - q^r} (1 - q^r + \alpha t + \beta t^2 + \cdots)
\]

and so the coefficient here is \( C_r \).

When \( n < r \), we desire the coefficient of \( t^{r-n-2} \) in

\[
-(q)_{r-1} C_n q^n (\Delta_{(n)} t^{-1} \phi(tq^n)) \frac{\phi(tq^k) \phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})}{\phi(tq) \phi(tq^2) \cdots \phi(tq^{n+1})} (q)_{n}
\]

which is just \( C_n r_{r-n}(r) \). Hence
and the determinental expression for $C_{N}$ is now obtained by applying Cramer's rule to the above system of equations with $0 \leq r \leq N$. This concludes the proof of Theorem 1.

Let us now show that $\lim_{q \to 1} \epsilon_{j}(r)(q)^{-1} = 0$ for $0 < j < r$.

\[
\lim_{q \to 1} \epsilon_{j}(r)(q)^{-1} = -\text{Coefficient of } t^{j-2} \text{ in } \left(\frac{r-1}{j}\right) \left[\frac{d}{dt} (t^{-1}\phi(t))\right] \phi(t)^{j-1}
\]

(2.3) 

\[
\begin{align*}
&= -\text{Coefficient of } t^{j-2} \text{ in } \frac{1}{j} \left(\frac{r-1}{j}\right)^{j-1} \left[\frac{d}{dt} (t^{-1}\phi(t))\right] \\
&= -\text{Coefficient of } t^{-1} \text{ in } \frac{1}{j} \left(\frac{r-1}{j}\right) \left[\frac{d}{dt} (t^{-1}\phi(t))\right] = 0,
\end{align*}
\]

since $dt'/dt = Ct^{-1}$ for any $r$ with $C \neq 0$ and since $(t^{-1}\phi(t))^{j} = \sum_{r=0}^{\infty} a_{r} t^{r}$.

(2.3) may now be applied to show that

\[
\lim_{q \to 1} \frac{C_{n}}{(q)_{n}} = \lim_{q \to 1} \frac{\delta_{n}}{(q)_{n}} = \left[n^{n-1} \left[\frac{d^{n-1}}{dt^{n-1}} (f'(t)\phi(t))^{n}\right]_{t=0}\right].
\]

Since $\epsilon_{j}(n) \equiv 0$ (due to the fact that $\Delta_{(0)}(t) \equiv 0$), we see that $\delta_{1} = c_{1}$ and so (2.4) is clearly true for $n = 1$. If we assume (2.4) to be valid up to, but not including a particular $n$, then, by (2.2),

\[
\frac{\delta_{n}}{(q)_{n}} = \sum_{j=1}^{n-1} \frac{C_{n-j}}{(q^{j+1})_{n-j}} \frac{\epsilon_{j}(n)}{(q)_{j}} + \frac{C_{n}}{(q)_{n}}.
\]

Consequently, by the induction hypothesis,

\[
\begin{align*}
\lim_{q \to 1} \frac{C_{n}}{(q)_{n}} &= \lim_{q \to 1} \frac{\delta_{n}}{(q)_{n}} - \sum_{j=1}^{n-1} \frac{(n-j)!}{n!} \left(\lim_{q \to 1} \frac{C_{n-j}}{(q)_{n-j}}\right) \left(\lim_{q \to 1} \frac{\epsilon_{j}(n)}{(q)_{j}}\right) \\
&= \lim_{q \to 1} \frac{\delta_{n}}{(q)_{n}} - \sum_{j=1}^{n-1} \frac{j!}{n!} \left[\frac{d^{n-j-1}}{dt^{n-j-1}} (f'(t)\phi(t))^{n-j}\right]_{t=0} \\
&= \lim_{q \to 1} \frac{\delta_{n}}{(q)_{n}} = n^{n-1} \left[\frac{d^{n-1}}{dt^{n-1}} (f'(t)\phi(t))^{n}\right]_{t=0}.
\end{align*}
\]

3. The Carlitz expansion (1.2). We now apply Theorem 1 to the special case $\phi(x) = 1 - ax$, $k = 1$. In this case, when $j < r$,
\[ \epsilon_j(r) = \frac{\frac{1}{r} q^{-j}}{(q)_{r-j}} \{ \text{Coefficient of } t^{i-2} \text{ in} \}
\]
\[ t^{-1}(1-atq)-t^{-1}q-r+i(1-atq^{-j+1})(atq^{r-j+2})_{j-1} \}
\]
\[ = \frac{\frac{1}{r} q^{-j}}{(q)_{r-j}} \{ \text{Coefficient of } t^{i-2} \text{ in} t^{-2}(1-q^{-r+i})(atq^{r-j+2})_{j-1} \}
\]
\[ = 0, \]
since the highest power of \( t \) in \( t^{-2}(1-q^{-r+i})(atq^{r-j+2})_{j-1} \) is \( t^{i-3} \). Hence, by Theorem 1, \( C_n = \delta_n \), and we have
\[ (3.1) \quad F(x) = F(0) + \sum_{n \geq 1} \frac{x^n}{(q)_n (aq)_n} \left[ \Delta_n^{-1}(\Delta F(t) \cdot (atq)_n) \right]_{t=0}. \]
When \( a = q^{-1} \) we get (1.2).

4. Evaluation of the \( C(n|q) \). We now treat the case originally envisioned by Carlitz; namely (1.3) where now \( F(x) = x, \phi(x) = (1-x)^{-1} \).

Theorem 2. Let \( C(n|q) \) be defined by (1.2), then
\[ q^{2n-2} \frac{1 - q}{1 - q^n} \binom{2n-2}{n-1} = C(n - 1|q) \]
\[ (4.1) \quad -q \sum_{j=1}^{n} (1 - q^{n-j})q^{n-j+1}(j-1) \binom{2j-1}{1-j} C(n - j|q). \]

Remark. This result exhibits clearly that \( C(n - 1|1) = n^{-1}(\sum_{n-1}^{2n-2}) \).

Proof. We assert that (4.1) is essentially (2.2) when \( \phi(x) = (1-x)^{-1}, k = 2 \). This is because in this case
\[ \delta_r = [\Delta_r^{-1}((\Delta r \cdot (tq^2)^{-1})_{r=0} = q^{2r-2}(1-q)(1-q^2r-2) \ldots (1-q^r), \]
and
\[ \epsilon_j(r) = -\frac{(q)_{r-j} q^{-j}}{(q)_{r-j}} \{ \text{Coefficient of } t^{i-2} \text{ in} \}
\]
\[ \frac{[1 - (1-tq)/q^{-i}]}{t^2(tq^{r-j+1})_{j}} \}(q)_{r-j}^{-1} \]
\[ = (q)_{r-j} \{ \text{Coefficient of } t^{i-2} \text{ in} \}
\]
\[ \frac{(1-q^r)(1-tq-tq^{-j+1})}{t^2(tq^{r-j+1})_{j+1}} \}(q)_{r-j}^{-1} \]
\[ = (q)_{r-j} \{ \text{Coefficient of } t^j \text{ in} \}
\]
\[ (1-q^{-j})(1-tq(1+q^{-r+j})) \sum_{n \geq 0} \binom{j+n}{n} q^n (r-j+1)n \}
\[ (q)_{r-j}^{-1} \]
\[ = (q)_{r-j} \{ \frac{(1-q^{r-j})(2j)}{q}(r-j+1) \}
\]
\[ -q(1-q^2(r-j))(2j-1) q^{r-j+1}(j-1) \}
\[ (q)_{r-j}^{-1} \]
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Hence, dividing (2.2) by \((q)_r\) and noting that \(C_n = C(N - 1|q)(q)_n\), we obtain

\[
q^r - 1 \frac{(1 - q)(2r - 2)}{1 - q^r} = \frac{C(r - 1|q)}{\binom{r}{q}}
\]

and noting that \(C(r - 1|q) = C(\binom{r}{q})\), we obtain

\[
C(r - 1|q) = \frac{1}{q} \sum_{j=1}^{r} \left(1 - q^{r-j}\right) q^\binom{r-j+1}{j-1} \frac{C(r-j+1|q)}{q^{r-j}(j-1)}
\]

as desired.

We can, of course, obtain a determinental expression for \(C(n - 1|q)\) as given by Theorem 1; however Theorem 2 seems to most clearly exhibit the relationship between \(C(n - 1|q)\) and the \(q\)-Catalan number \(\binom{n}{1}_q^{-1} (2n-2)_q^2\).

REFERENCES


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