ON THE NUMBER OF GENERATORS
OF POWERS OF AN IDEAL

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ABSTRACT. Let \( I \) be an ideal of a quasi-local ring. In this note we consider the question of how small—in terms of numbers of generators—the powers of the ideal \( I \) can be.

In this note all rings are commutative with identity. If \( I \) is an ideal of a ring, \( v(I) \) denotes the minimal number, which may be infinite, of generators of \( I \).

Proposition 1. Let \( I \) be an ideal of a quasi-local ring \( A \). If, for some integer \( t \geq 1 \), \( v(I^t) = 1 \), then either \( v(I^k) = 1 \) for all positive integers \( k \) or \( I \) consists of zero divisors. If, for some integer \( t > 1 \), \( v(I^t) = 2 \), then either \( v(I^k) = 2 \) for all positive integers \( k \) or \( I \) consists of zero divisors.

We will use the following lemmas in the proof of the proposition.

Lemma 2. Let \( I \) be an ideal of a ring \( R \). Let \( J \) and \( K \) be subsets of \( I \) such that \( J \subseteq I \) and \( K \subseteq I^{t-1} \) for some positive integer \( t \). If \( I^t = JK \), then \( I^{t+1} = J^2K \).

Proof. \( I^{t+1} = JK = JJK \subseteq J^2K = J^2K \).

It is an immediate consequence of Lemma 2 that if some power \( I^t \) of an ideal \( I \) is finitely generated then so is every higher power. Since \( I^t \) is generated by sums of monomials of degree \( t \) in elements of \( I \), take \( J \) to be the ideal generated by all the elements of \( I \) which appear in the monomials in such a generating set. Then \( J \) is a finitely generated ideal contained in \( I \), \( I^t = J^t \) and, by Lemma 2, \( I^{t+k} = J^{t+k} \) for \( k \geq 0 \).

If an element \( x \) in \( I \) appears in \( j \) monomials in some minimal generating set for \( I \) we will say that \( I^t \) has an overlap of length \( j \). It follows from Lemma 2 that if \( v(I^t) = n \) and if \( I^t \) has an overlap of length \( n \), then \( v(I^{t+k}) \leq n \) for \( k \geq 0 \). (In particular, if an ideal \( I \) in a quasi-local ring has some power principal, then every higher power is principal.)

If \((A, m)\) is a quasi-local ring and \( u \) an indeterminate we will often make use of the faithfully flat change of rings \( A \to A(u) = A[u]_mA[u] \).

Received by the editors September 13, 1974.


Key words and phrases. Quasi-local ring, ideal, faithfully flat ring extension.

1 The author received partial support from the National Science Foundation.
Lemma 3. Let \((A, m)\) be a quasi-local ring and \(I\) a finitely generated ideal. Let \(t\) be any integer > 1. Then either \(I^t\) is principal or \(I^tA(u)\) has an overlap of length 2.

Proof. Let \(I = (x_1, \ldots, x_n)\) and suppose that \(v(I^t) > 1\), say

\[ I^t = (x_{i_1}^{\alpha_1} \ldots x_{i_r}^{\alpha_r}, \ldots, x_{j_1}^{\beta_1} \ldots x_{j_s}^{\beta_s}) \]

with \(\alpha_i, \beta_i\) positive integers such that \(\alpha_1 + \cdots + \alpha_r = \beta_1 + \cdots + \beta_s = t\). Assume that every \(x_i\) appears in at most one monomial in this generating set for \(I^t\). Now

\[ x_{i_1}^{\beta_1-1} x_{j_1}^{\beta_1} x_{i_2}^{\beta_2} \ldots x_{j_s}^{\beta_s} = ax_{i_1}^{\alpha_1} \ldots x_{i_r}^{\alpha_r} + \cdots + bx_{j_1}^{\beta_1} \ldots x_{j_s}^{\beta_s} \]

for some \(a, \ldots, b \in A\). If one of the coefficients \(a, \ldots, b\) is not in \(m\) we can replace the generator it multiplies by \(x_{i_1}^{\beta_1-1} x_{j_1}^{\beta_1} x_{i_2}^{\beta_2} \ldots x_{j_s}^{\beta_s}\) and produce an overlap in \(I^t\) itself. So we assume that all the coefficients are in \(m\).

We pass to the ring \(A(u)\) and change the basis of \(IA(u)\) as follows:

\[ x_{i_1} = y_{i_1} + uy_j; \quad x_k = y_k \quad \text{for} \quad k \neq i_1. \]

Then

\[ IA(u) = (y_1, \ldots, y_n) \quad \text{and} \quad I^tA(u) = (y_{i_1}^{\alpha_1} \ldots y_{i_r}^{\alpha_r}, \ldots, y_{j_1}^{\beta_1} \ldots y_{j_s}^{\beta_s})A(u), \]

for

\[ I^t \subseteq (y_{i_1}^{\alpha_1} \ldots y_{i_r}^{\alpha_r}, \ldots, y_{j_1}^{\beta_1} \ldots y_{j_s}^{\beta_s})A[u] + uI^tA[u]. \]

Now (*) gives

\[ y_{i_1}^{\beta_1-1} y_{j_1}^{\beta_1} y_{i_2}^{\beta_2} \ldots y_{j_s}^{\beta_s} + (u-b) y_{j_1}^{\beta_1} \ldots y_{j_s}^{\beta_s} \in mI^tA(u). \]

Hence, if we substitute \(y_{i_1}^{\beta_1-1} y_{j_1}^{\beta_1} y_{i_2}^{\beta_2} \ldots y_{j_s}^{\beta_s}\) for the generator \(y_{j_1}^{\beta_1} \ldots y_{j_s}^{\beta_s}\) of \(I^tA(u)\), we have that \(I^tA(u)\) has an overlap of length 2.

Corollary. Let \(A\) be a quasi-local ring and \(I\) an ideal such that for some \(t > 1, v(I^t) = 2\). Then \(v(I^{t+k}) \leq 2\) for all \(k \geq 0\).

Proof. By Lemma 2 we may assume that \(I\) is finitely generated. \(I^tA(u)\) has 2 generators and, by Lemma 3, an overlap of length 2. Hence \(v(I^{t+k}A(u)) \leq 2\) and \(v(I^{t+k}) \leq 2\).

Proof of Proposition 1. If \(I\) is not principal, let \(t > 1\) be the least integer such that \(v(I^t) = 1\). Say \(I^t = (x_1^{\alpha_1} \ldots x_r^{\alpha_r})\), where the \(\alpha_i\) are positive integers such that \(\alpha_1 + \cdots + \alpha_r = t\) and the \(x_i\) are in \(I\). Since it is enough to show that some \(x_i\) is a zero divisor we may assume that \(I = (x_1, \ldots, x_r)\) and that \(x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_r^{\alpha_r}\) is one of the generators in a minimal basis for \(I^{t-1}\). Let \(z\) be another such generator. Then the relation \(x_1z = bx_1^{\alpha_1} \ldots \)
$x^*_r$, for some $b$ in $A$, shows that $x_1$ is a zero divisor. To prove the second statement, suppose that $t > 1$ is a positive integer such that $\nu(l^t) = 2$ and $\nu(l^{t-1}) > 2$. By passing to the ring $A(u)$ we may assume that $l^t$ has a minimal generating set of the form $x_1^a x_2^a \cdots x_r^a x_{r+1}^{a_{r+1}} \cdots x_s^{a_s}$ with $x_1, \ldots, x_s \in l$ and $a_j$ positive integers such that $a_1 + \cdots + a_r = a_{r+1} + \cdots + a_s = t - 1$. Then there is a $z$ in $l^{t-1}\setminus(x_1^a x_2^a \cdots x_r^a x_{r+1}^{a_{r+1}} \cdots x_s^{a_s})$. The relation

$$z x_1 = a x_1^a x_2^a \cdots x_r^a x_{r+1}^{a_{r+1}} \cdots x_s^{a_s} + b x_1^a x_2^a \cdots x_r^a x_{r+1}^{a_{r+1}} \cdots x_s^{a_s},$$

for some $a, b$ in $A$, shows that $x_1$ is a zero divisor and that $l A(u)$ consists of zero divisors. Hence (cf. [3]) $l$ consists of zero divisors.

The following example was shown to me by P. Eakin and W. Heinzer. (The reference [1] was supplied by the referee.) The example demonstrates that "very big" ideals in quasi-local domains can have powers generated by 3 elements. Let $n \in \{1, 2, \ldots, \infty\}$. Let $S$ be any ring having an ideal $J$ minimally generated by $n$ elements. Let $R = S[[x^4, x^5, Jx^{11}]]$, where $x$ is an indeterminate, and consider the ideal $l = (x^8, x^9, Jx^{11})R$. $\nu(l) = n + 2$, but $\nu(l^2) = 3$ since $l^2 = (x^8, x^9, x^{10})R$. Variation of the semigroup used as exponents for $x$ leads to many such examples. Notice, however, that the domain $R$ has dimension greater than one. What happens in the one-dimensional local case, except for some very special ideals (e.g. [2], [4]), is an open question. It is possible that the following generalization of Lemma 3 holds: if $l$ is an ideal of a quasi-local ring $A$ with $\nu(l^t) = k$ where $t > k$, then $\nu(l^{t+j}) \leq k$ for all $j \geq 0$. The combinatorial methods used above give a rather lengthy verification of this for $k = 3$, but this does not seem to be a good way to attack the general case.

* Added in proof. This conjecture has been proved by Eakin and Sathaye, Prestable ideals, J. Algebra (to appear). Also pertinent are the papers A note on the Hilbert function of a one-dimensional Cohen-Macaulay ring (preprint) by Herzog and Waldi, and Hilbert functions of graded algebras (preprint) by Stanley.

REFERENCES


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