POWER-ASSOCIATIVITY OF ANTIFLEXIBLE RINGS

HASAN A. ÇELİK AND DAVID L. OUTCALT

ABSTRACT. Conditions which force an antiflexible ring of characteristic $p$ to be power-associative are determined.

1. Introduction. We prove that antiflexible rings are always $m$th power-associative except when $m - p$ for $r \in \mathbb{Z}^+$, where the prime $p > 2$ is the characteristic of $R$. It is proved that $R$ is power-associative provided there exist some $i, j \in \{1, 2, \ldots, p^r - 1\}$ such that $(|i - j|, p) = 1$ and $x^{p^r - i}x^i = x^{p^r - 1}x^i$ for $r \in R$ and $r \in \mathbb{Z}^+$. We show that this condition may not be omitted by constructing a family of antiflexible rings such that $x^{p^r - 1}x = 0$ but $xx^{p^r - 1} \neq 0$. In particular, it is possible to construct antiflexible rings with elements which are right nilpotent but not left nilpotent.

2. Preliminaries. A ring is antiflexible provided
\[(2.1) (x, y, z) = (z, y, x)\]
for all $x, y, z$ in the ring where the associator $(a, b, c)$ is defined by $(a, b, c) = (ab)c - d(bc)$. Let $x$ be an element of a ring. We define $x^n$ for all positive integers $n$ by
\[(2.2) x^1 = x; \quad x^k = x^{k-1}x, \quad k = 2, 3, 4, \ldots.\]
A ring is $m$th power-associative provided
\[(2.3) (x^i, x^j, x^k) = 0\]
for all $x$ in the ring and for all positive integers $i, j, k$ such that $i + j + k \leq m$, $m$ a positive integer. This is equivalent to saying that $x^{i+j} = x^i x^j$ for all $i, j$ such that $i + j \leq m$. A ring is power-associative provided it is $m$th power-associative for every positive integer $m$. The following two identities hold in any ring:
\[(2.4) (w, x, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z,\]
\[(2.5) [x, y, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y)\]
where the commutator $[a, b]$ is defined by $[a, b] = ab - ba$.

Let $k$ be a positive integer. If a ring is of characteristic $k$, then $kx = 0$ for all $x$ in the ring, and if $n$ is prime to $k$, then $nx = 0$ implies $x = 0$ for $x$ in the ring; and if a ring is of characteristic zero, then $nx = 0$ implies $x = 0$.
for every positive integer \( n \) and for \( x \) in the ring.

From here on, \( R \) will denote an antiflexible ring of characteristic prime
to 2 which is also third power-associative. Hence
\[
(x, x, x) = 0
\]
for all \( x \in R \). Linearization of (2.6) yields
\[
(x, y, z) + (y, z, x) + (z, x, y) = 0
\]
upon application of (2.1) and the fact that \( R \) is of characteristic prime to 2.
Subtracting (2.7) from (2.5) and using (2.1), we obtain
\[
(x, y, z) - x[y, z] - [x, z]y = -2(x, z, y).
\]
Set \( y = x \) or \( z = x^2 \) in (2.8). Then \( (x, x^2, x) = 0 \) and hence
\[
(x, x^2, x) = 0 = (x^2, x, x)
\]
from (2.7) with \( y = x \) and \( z = x^2 \) and (2.1). Therefore, \( R \) is 4th power-associative as
was first established by Kosier in [3]. Thus, as Kosier observed,
it follows from a theorem of Albert [1] that if \( R \) is of characteristic 0 then \( R \)
is power-associative. On the other hand, Rodabaugh published an example
[5] showing that if \( R \) is not of characteristic 0 then \( R \) need not be power-associative. Theorems of power-associativity of arbitrary rings and algebras
have appeared in Albert [1], Kokoris [2], Leadley and Richie [4].

3. Main section. We make use of the following result of Albert [1].

**Lemma 1.** Let the characteristic of a ring \( A \) be prime to two, \( n \geq 4 \),
\[ x^{\lambda}x^\mu = x^{\lambda+\mu} \quad \text{for} \quad \lambda + \mu < n. \]
Then
\[ n[x^{n-1}, x] = 0, \quad [x^{n-\alpha}, x^\alpha] = \alpha[x^{n-1}, x], \quad \alpha \in \{1, \ldots, n-1\}, \]
so that, if \( n \) is prime to the characteristic of \( A \), we have \( x^{n-\alpha}x^\alpha = x^\alpha x^{n-\alpha} \)
for \( \alpha \in \{1, 2, \ldots, n-1\} \).

Recall that by \( R \) we always mean an antiflexible ring of characteristic
prime to 2 which is also third power-associative. By \( R[x] \) we mean the sub-
ring of \( R \) generated by \( x \in R \).

**Lemma 2.** If \([x^{m-1}, x] = 0\), then \( R[x] \) is \( m \)th power-associative provided
\( R[x] \) is \((m - 1)\)st power-associative.

**Proof.** Using (2.8),
\[
0 = [x^{m-1}, x] = [x^{m-1-i}x^i, x] = -2(x^{m-1-i}, x, x^i)
\]
Hence,
\[
x^{m-i}x^i = x^{m-1-i}x^{i+1}, \quad 1 \leq i \leq m - 2,
\]
which implies that \( x^{m-k}x^k = x^m \) for all \( 1 \leq k \leq m - 1 \) and hence \( R[x] \) is
\( m \)th power-associative.
Lemma 3. For and \( m \in \mathbb{Z}^+ \), if \( R[x] \) is \((m-1)st\) power-associative then

\[
(3.2) \quad (i+1)x^m = 2x^{m-i}i^i + (i-1)xx^{m-1}, \quad 1 \leq i \leq m-1.
\]

Proof. We establish the identity by induction. \( i = 1 \) case is true by definition. Suppose that the identity holds for \( i = k-1 \). Then

\[
x^{m-k+1}x^{k-1} = (x^{m-k}, x, x^{k-1}) + x^{m-k}xx^k = (xx^{m-k-1}, x, x^{k-1}) + x^{m-k}xx^k
\]

since \( R[x] \) is \((m-1)st\) power-associative. By (2.4), we obtain, using (2.1) and (2.7),

\[
(xx^{m-k-1}, x, x^{k-1}) = (x, x^{m-k}, x^{k-1}) - (x, x^{m-k-1}, x^k)
\]

\[
= -(x^{m-k}, x, x^{k-1}) - (x^{m-k}, x^{k-1}, x) - (x, x^{m-k-1}, x^k)
\]

\[
= -x^{m-k+1}x^{k-1} + x^{m-k}xx^k - x^{m-1}x + x^{m-k}xx^k + x^{m-k}x^k + xx^{m-1}.
\]

Thus,

\[
x^{m-k+1}x^{k-1} = -x^{m-k+1}x^{k-1} + 2x^{m-k}xx^k - x^m + xx^{m-1}
\]

or

\[
(3.3) \quad 2x^{m-k+1}x^{k-1} = 2x^{m-k}xx^k - x^m + xx^{m-1}.
\]

By the induction hypothesis,

\[
kx^m = 2x^{m-k+1}x^{k-1} + (k-2)xx^{m-1} = 2x^{m-k}xx^k - x^m + (k-1)xx^{m-1}
\]

using (3.3), hence

\[
(k+1)x^m = 2x^{m-k}xx^k + (k-1)xx^{m-1}.
\]

Theorem 1. Let the characteristic of \( R \) be a prime \( p \). Then \( R \) is \( p \)th power-associative if and only if for all \( x \in R \) there exists some \( i, j \in \{1, 2, \ldots, p-1\} \), \( i \neq j \) such that \( x^p-i^i = x^p-j^i \).

Proof. One direction is obvious. Assume that \( x^p-i^i = x^p-j^i \) for some \( i, j \in \{1, 2, \ldots, p-1\} \), \( i \neq j \). By Lemma 1, \( R \) is \((p-1)st\) power-associative. Then we may use (3.2):

\[
(i+1)x^p = 2x^{p-i^i} + (i-1)xx^p, \quad (j+1)x^p = 2x^{p-j^j} + (j-1)xx^p.
\]

Subtraction yields \((i-j)x^p = (i-j)xx^p-1\) since \( x^p-i^i = x^p-j^j \). However, \(|i-j| \in \{1, 2, \ldots, p-1\} \), hence \( x^p = xx^{p-1} \). Thus \( R \) is \( p \)th power-associative by Lemma 2.

Theorem 2. Let the characteristic of \( R \) be a prime \( p > 2 \). Then \( R \) is power-associative if and only if for every \( x \in R \), \( x^p-i^i = x^p-j^j \) for all \( r \in \mathbb{Z}^+ \) and for some \( i, j \in \{1, 2, \ldots, p-1\} \) with \(|i-j|, p \) = 1.

Proof. The case \( r=1 \) yields that \( R \) is \( p \)th power-associative. Since any integer \( m \) between \( tp \) and \((t+1)p \) for \( t \in \mathbb{Z}^+ \) is relatively prime to \( p \),
Lemma 1 implies that $R$ is $m$th power-associative providing $R$ is $t$th power-associative.

Next, we establish that $R$ is $t$th power-associative for $t \in \{1, 2, \ldots, p - 1\}$. Using induction on $t$, we may assume that $R$ is $(t - 1)$th power-associative. Hence $R$ is also $(tp - 1)$th power-associative. So, for all $x \in R$, we have

$$x^{t(p - 1)} = x^{tp - t} = (x^t)^{p - 1}.$$ 

Therefore,

$$x^{tp - t}x^t = x^{(t - 1)p - 1}x^t = x^t(x^{p - 1}) = xtx^{ip - t};$$

hence $[x^{tp - t}, x^t] = 0$. Now Lemma 1 yields the claim.

We may now assume that $R$ is $(p' - 1)$th power-associative and prove that it is $p'$th power-associative. Let $i, j \in \{1, \ldots, p' - 1\}$ with $i \neq j$ and $\gcd(i - j, p') = 1$, and $x^{p' - i}x^i = x^{p' - i}x^i$. Using (3.2),

$$(i + 1)x^{p'} = 2x^{p' - i}x^i + (i - 1)x^{p' - 1},$$

$$(j + 1)x^{p'} = 2x^{p' - i}x^i + (j - 1)x^{p' - 1},$$

which yield $x^{p'} = xx^{p' - 1}$. This completes the proof.

We close with an example to show that Theorem 2 is the best possible for antiflexible rings. Rodabaugh's example [5] shows that there exist third power-associative, antiflexible rings of characteristic $p$, $p$ an odd prime, which are not $p$th power-associative. Our example goes one step further. It shows that if $R$ is of characteristic $p$, $p$ a prime, then it is possible for $x^{p' - i}x^i = 0$ for some $i$ while $x^{p' - i}x^i \neq 0$ for all $j \neq i$. Furthermore, the example provides antiflexible algebras of dimension $p'$, for $r \in \mathbb{Z}^+$, which are not $p'$th power-associative and which contain elements which are right nilpotent but not left nilpotent.

**Example.** Let $F$ be a field of characteristic $p > 3$, $p$ a prime. Let $B_p^r$ be the $p'^r$th dimensional algebra over $F$ with basis $1, x, x^2, \ldots, x^{p' - 1}$, where $r \in \mathbb{Z}^+$ and

1. $x^{k+l} = x^k + l$ if $2 \leq k + l \leq p' - 2$,
2. $x^{k+l} = \frac{1}{2}(1 - l)a$, $\alpha \in F, \alpha \neq 0$, if $k + l = p'$,
3. $x^{k+l} = \frac{1}{2}\alpha x^{k+l-p'}$ if $k + l > p'$, $k, l \in \{1, 2, \ldots, p' - 1\}$.

Observe that $x^{p'} = 0$, and $xx^{p' - 1} = \alpha \neq 0$.

To show that $B_p^r(\alpha)$ is a third power-associative antiflexible algebra one should verify that

1. $k + l + m = p'$.

It is immediate that $(x^k, x^l, x^m) = (x^m, x^l, x^k)$ for all $k, l, m \in \{1, 2, \ldots, p' - 1\}$. Identity (3.4) includes combinations of the following cases:

1. $k + l + m = p'$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(ii) \( k + l = p^r, \ l + m = p^r, \ k + m = p^r; \)
(iii) \( k + l > p^r, \ l + m > p^r, \ k + m > p^r, \ k + l + m = 2p^r, \ k + l + m < 2p^r, \)
\( k + l + m > 2p^r, \)

We illustrate one typical case:

\[
\begin{align*}
\kappa + \lambda > p^r & \quad \text{and} \quad \kappa + \lambda + m = 2p^r, \\
\kappa + \lambda + m > 2p^r, \\
\lambda + m > 2p^r \\
x^k x^l y = v^a (\xi + \eta) \\
x^k x^l y = v^a (\xi + \eta) \\
\end{align*}
\]

Thus,

\[
(x^k, x^l, x^m) - (x^m, x^l, x^k) = 4a^2[(1 - m) - (1 + k) - (1 - k) + (1 + m)] = 0.
\]

(3.4) can similarly be verified for all cases of (i), (ii), and (iii).

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY, POMONA, CALIFORNIA 91768

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106