STABILITY THEORY FOR HILL EQUATIONS
WITH GENERALIZED COEFFICIENT

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ABSTRACT. A simple geometric proof is given for the existence of Ljapunov's intervals of stability and instability for Hill equations with generalized coefficients.

The basic theorem of Ljapunov on the stability of Hill equations

\[ x'' + \lambda p(t)x = 0, \quad p(t + T) = p(t), \quad p' \]

is:

\( (L) \) There exists a sequence of eigenvalues

\[ \lambda''_{-1} < \lambda'_0 \leq \lambda''_0 < \cdots < \lambda'_k = \lambda''_k < \cdots \]

of (1) for periodic boundary value problems

\[ x(t + T) = (-1)^k + 1x(t), \quad x'(t + T) = (-1)^k + 1x'(t). \]

The intervals of stability of (1) are \([\lambda''_1, \lambda'_1]\) and the intervals of instability \([\lambda'_1, \lambda''_1]\) (except if \(\lambda'_k = \lambda''_k\) when all solutions of \(x + \lambda''_k p x = 0\) are periodic).

A complete proof of the corresponding theorem for

\( (2) \)

\[ x'' + (\lambda - q(t))x = 0 \]

and continuous \(q(t)\) is given by Magnus and Winkler [4]. In the case of (2), no hypothesis is needed on the sign of \(q\). The proof of [4] is laborious; it is classified as "very complicated" in [3]. In the present note, we give a proof of (L), based on elementary geometry, Sturm-Liouville and Floquet theories, that is short and easy for \(p\) or \(q\) continuous functions. With the few additional arguments presented here, the proof is valid if

(A) \(p(t)\) is periodic of period \(T\),

(B) \(p(t)\) is the generalized derivative of a bounded, monotone increasing, one-sided continuous function and there exists \(\epsilon > 0\) such that

\( (B_1) \)

\[ \int_{t_0}^{t_0 + \Delta} p(t) \, dt > \epsilon \Delta \]

for all \(t_0\) and \(\Delta\).

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The monotonicity condition would be superfluous for (2). If (B) holds, (1) (or (2)) is equivalent [1, (2)] to a linear system with continuous coefficients and, therefore, the initial value problem has unique solutions. We denote by \( x''(t) \) the curve \((x''_1(t), x''_2(t))\) whose coordinate functions are solutions of (1) for the standard initial conditions \( x''(u) = (1, 0), x''(u) = (0, 1) \). The curve \( x''(t) \) is locally convex, its angular velocity is positive, and its Wronskian \( W \) is unity. The solutions of (1) have forward and backward derivatives. If \( p(t) \) is the generalized derivative of \( Q(t) \), then \([1, (9)]\)

\[
x'(t) = x''(t) - \Delta Q(t)x(t)
\]

where \( \Delta Q(t) = Q(t + 0) - Q(t - 0) \). Every scalar solution is differentiable at its zeros. The conjugate point \( c(u, \lambda) \) is the smallest value \( t > u \) for which \( x''(t) \) is linearly dependent on \( x''(u) \). The \( k \)th conjugate point is \( c_k(u, \lambda) = c(c_{k-1}(u, \lambda), \lambda) \) where \( c = c_0 \). The conjugate points are the zeros of \( x''_u \) and for the angle

\[
\alpha(t) = \tan^{-1}\frac{x''_2(t)}{x''_1(t)}
\]

we have

\[
\alpha(c_k(u, \lambda)) = \alpha(u) + (k + 1)\pi.
\]

The function \( c_k(u, \lambda) \) is differentiable in both variables [2, pp. 225–226], and for \( x_i(u) = 0 \) we have \([2, (8)]\)

\[
\frac{\partial c}{\partial u} = x'_i(u)^2/x'_i(c(u, \lambda))^2.
\]

The independent variable \( t \) is two times the area covered by the vector \( x(t) \) (up to an additive constant).

**Lemma 1.** The function \( \alpha'(t) \) is strictly increasing with \( \lambda \), \( c(u, \lambda) \) is strictly decreasing with \( \lambda \).

The lemma is simply Sturm’s comparison theorem which is valid in our case \([2, \text{Theorem 1}]\). In fact, \( \alpha' = (x'_1^2 + x'_2^2)^{-1} \), and for identical initial conditions the comparison theorem implies that for larger \( \lambda \) the radius \( |x| \) is smaller and, since \( t \) is the area, the angular velocity must be greater. The second assertion is an immediate consequence of the first.

**Lemma 2.** The problem \( x'' + \lambda p(t)x = 0, x(u) = x(u + T) = 0 \), has a discrete spectrum of eigenvalues \( \lambda_k(u) \) of all orders, \( \lambda_k(u) \to \infty \).

The index of the eigenvalue is the number of zeros of an eigenfunction for \( u \leq t < u + T \). For the vector \( x'' \),

\[
\alpha(u + T) = (k + 1)\pi.
\]

By Lemma 1, there can be no more than one eigenvalue to an index. By the
Sturm comparison theorem and (B1), there exist eigenvalues of all orders. The remainder of the lemma is proved as for continuous coefficients.

Lemma 2 will not hold for equation (1) without condition (B1). For example, for \( u = 0 \) and \( p(t) \) a sum of \( \delta \)-functions of mass 1 at \((2n + 1)T/2\), only \( \lambda_0 \) will exist.

The eigenvalues can also be defined by

\[
(7) \quad c_k(u, \lambda_k(u)) = u + T
\]

and, hence, are differentiable functions of \( u \) [2, (10)].

**Definition [5].** \( \lambda_k' = \min \lambda_k(u) \), \( \lambda_k'' = \max \lambda_k(u) \). It is sufficient to consider an interval \( 0 \leq u < T \). By (7),

\[
\frac{\partial c_k}{\partial u} + \frac{\partial c_k}{\partial \lambda} \frac{d \lambda_k}{d u} = 1
\]

and for both \( x^u(t, \lambda_k') \) and \( x^u(t, \lambda_k'') \) we have

\[
x^u_2(u + T) = (-1)^{k+1} x^u_2(u) = 0, \quad x^{u'}_2(u + T) = (-1)^{k+1} x^{u'}_2(u).
\]

The sign follows from (6) and \( W = 1 \).

We now use Floquet theory (e.g. [4, §5.2]). If \( \lambda(u) \) is a solution of a problem of Lemma 2, it follows from (6) that the matrix that maps \( (x^u(u), x^{u'}(u)) \) onto \( (x^u(u + T), x^{u'}(u + T)) \) has a real eigenvalue and \( \lambda(u) \) is unstable for (1). If there is not an eigenvalue for any \( u \) and \( k \), no such matrix will have a real eigenvalue and \( \lambda \) is stable. Hence, all we have to prove is that the \( \lambda_k' \) and \( \lambda_k'' \) interlace in the right order.

**Lemma 3.** \( \lambda_k(u) > \lambda_{k-1}(v) \) for all \( u \) and \( v \).

The arc \( x^u(t) \) intersects the \( x_1 \)-axis at \( c_i(u, \lambda_k(u)) \), \( i = 0, 1, \ldots \). We estimate \( c_{k-1}(u, \lambda_k(u)) \). Since everything is periodic, we can assume \( u \leq v < u + T \); more specifically, we assume \( c_{j-1}(u, \lambda_k(u)) \leq v < c_j(u, \lambda_k(u)) \) and put \( u = c_{j-1}(u, \lambda_k(u)) \). The arc \( x^u \) intersects the line \( Ox^u(v) \) once between two consecutive conjugate points. Hence,

\[
c_{j+k-1}(u, \lambda_k(u)) \leq c_{k-1}(v, \lambda_k(u)) < c_{j+k}(u, \lambda_k(u)).
\]

Since \( c_{j+k}(u, \lambda_k(u)) = c_{j-2}(u + T, \lambda_k(u)) \), we have

\[
c_{k-1}(v, \lambda_k(u)) - v < c_{j+k}(u, \lambda_k(u)) - c_{j-1}(u, \lambda_k(u)) = [c_{j+k}(u, \lambda_k(u)) - (u + T)] + [u + T - c_{j-1}(u, \lambda_k(u))] = u + T - c_{j-1}(u, \lambda_k(u)) + c_{j-2}(u, \lambda_k(u)) - u < T = c_k(u, \lambda_k(u)) - u.
\]
Since $c_{k-1}(v, \lambda_{k-1}(v)) - v = T$, Lemma 3 follows from Lemma 1. We have proved

**Theorem.** (L) holds under the hypotheses (A), (B).

**REFERENCES**


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