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ON A THEOREM OF MENCHOFF

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ABSTRACT. Construction of a continuous real-valued function which disagrees almost everywhere with any absolutely convergent Fourier series.

A classical theorem of Menchoff states that if f is a finite measurable function on the circle T and if $\epsilon > 0$, there exist functions g on T with uniformly convergent Fourier series and such that $f(t) = g(t)$ except possibly on a set of measure less than ϵ. The question has been raised (e.g. [1, problem 7, p. 527]) whether the theorem can be strengthened to yield, for any f and ϵ as above, an *absolutely convergent* Fourier series whose sum $g(t)$ agrees with $f(t)$ except for a set of t of measure less than ϵ. Our purpose in this note is to show that this cannot be done; in fact we prove more:

Theorem. *There exists a continuous real-valued function f on T such that if g is the sum of an absolutely convergent Fourier series then $\{t; f(t) = g(t)\}$ has measure zero.*

One can sharpen the problem by imposing on f a modulus of continuity; in fact one may conjecture that there exists an $f \in \mathrm{Lip}\, 1/2$ which differs from any absolutely convergent Fourier series almost everywhere. The construction we give below yields a function whose modulus of continuity is bounded by $(-\log h)^{-1}$.

We denote by $A(T)$ the Banach algebra of absolutely convergent Fourier series. For a closed set $E \subset T$, we denote by $A(E)$ the Banach algebra of restrictions to E of the elements of $A(T)$, the norm being the standard (quotient) norm.

The basic tool is the so-called Rudin-Shapiro sequence $\{\epsilon_n\}$, $\epsilon_n = \pm 1$, $n = 0, 1, 2, \ldots$, which has the property that for all positive integers N, M

$$\left\| \sum_{N+1}^{N+M} \epsilon_n e^{int} \right\|_{C(T)} \leq 5\sqrt{M} \tag{1}$$

(cf. [2, pp. 33–34]).

Define for $x \geq 0$,

$$\Phi(x) = \begin{cases} \epsilon_n, & n \leq x \leq n + \tfrac{1}{2}, \\ \text{linear for } n + \tfrac{1}{2} \leq x \leq n + 1, \end{cases} \tag{2}$$

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(3) $$\phi_k(t) = \Phi(8^k t/2\pi), \qquad 0 \le t \le 2\pi.$$

One checks in the construction of $\{\epsilon_n\}$ that $\epsilon_0 = \epsilon_{8^k} = 1$ for all k and, thus, $\phi_k(0) = \phi_k(2\pi) = 1$, and we continue ϕ_k by periodicity or consider it as a function on $T = R/2\pi Z$.

We now write $k_m = m!$ and $a_m = k_m^{-1}$, notice that

(4) $$a_m^{-1} 2^{3k_{m-1} - k_m} = o(1), \quad a_m^{-1} a_{m+1} = o(1), \quad a_m k_m = 1,$$

and claim that

(5) $$f = \sum a_m \phi_{k_m}$$

satisfies the condition imposed in the theorem. The proof uses the following simple

Lemma. *Let k be a positive integer, l and n positive integers such that $l + n \le 8^k$. Assume that $g \in C(T)$ and for some real-valued c and some α, $0 \le \alpha \le \frac{1}{2}$, we have*

(6) $$|g(t) - \phi_k(t) - c| < 1/10$$

on 90% of the points of E where $E = \{2\pi(\alpha + j)8^{-k}\}_{j=l}^{l+n}$. Then $\|g\|_{A(E)} > \pi^{-1}\log n - 4$.

Proof. Denote $\mu = \sum_l^{l+n} \epsilon_j \delta_{2\pi(\alpha+j)8^{-k}}$, that is, the measure carried by E whose mass at $2\pi(\alpha + j)8^{-k}$ is equal to ϵ_j. By (1) we have $\|\hat{\mu}\|_\infty \le 5\sqrt{n}$.

Select a real number c_1 so that $c_1 = c$ (mod 4) and $|c_1| \le 2$. By (6), we have

$$\int \sin \frac{\pi}{2}(\mathrm{Re}\; g - c_1)\, d\mu > \frac{7}{10} n.$$

Thus,

$$\frac{7}{10} n \le \left\| \sin \frac{\pi}{2}(\mathrm{Re}\; g - c_1) \right\|_{A(E)} \|\hat{\mu}\|_\infty \le 5\sqrt{n} \left\| \sin \frac{\pi}{2}(\mathrm{Re}\; g - c_1) \right\|_{A(E)},$$

and

$$\frac{7}{50}\sqrt{n} \le \left\| \sin \frac{\pi}{2}(\mathrm{Re}\; g - c_1) \right\|_{A(E)} \Rightarrow \|g\|_{A(E)} \ge \frac{2}{\pi} \log \frac{7\sqrt{n}}{50} - |c_1|,$$

that is

$$\|g\|_{A(E)} \ge \pi^{-1} \log n - (2 + 2\pi^{-1} \log(50/7));$$

and the lemma follows.

We now turn to the proof that the function f defined by (5) satisfies the condition stated in the theorem.

Let $g \in A(T)$ and assume that $G = \{t;\, g(t) = f(t)\}$ has positive measure. Let t_0 be a point of density of G. For all sufficiently large m, the relative measure of G in $I_m = (t_0 - 2\pi 2^{-k_m}, t_0 + 2\pi 2^{-k_m})$ is greater than 99%.

It is clear that for an appropriate α_m, $0 \le \alpha_m \le \frac{1}{2}$, G contains at least 98% of the points of the form $2\pi(\alpha_m + j)8^{-k_m}$ which are contained in I_m. We denote the set of all these points by E_m. E_m is an arithmetical progression of length $2 \cdot 4^{k_m}$.

We now write

$$f = \sum_{1}^{m-1} a_j\phi_{k_j} + a_m\phi_{k_m} + \sum_{m+1}^{\infty} a_j\phi_{k_j}$$

and notice that the first sum differs from a constant on I_m by at most 2^{2-k_m} $\sum_1^{m-1} a_j 8^{k_j} << 2^{3k_m - 1 - k_m}$ (the half length of I_m being $2\pi 2^{-k_m}$ and the maximum slope of the sum is bounded by $(2\pi)^{-1}4\sum_1^{m-1} a_j 8^{k_j}$).

On the other hand, $\sum_{m+1}^{\infty} a_j\phi_{k_j}$ is uniformly bounded by $2a_{m+1}$. Thus

$$a_m^{-1}(g - g(t_0)) = \phi_{k_m} + \text{const} + o(1)$$

on more than 90% of the points of E_m and it follows by the lemma that

$$\|a_m^{-1}(g - g(t_0))\|_{A(E_m)} \ge \pi^{-1}k_m$$

and so

$$\|g - g(t_0)\|_{A(I_m)} > \pi^{-1}a_m k_m = \pi^{-1}.$$

On the other hand, a classical theorem of Wiener [3, p. 88] implies that $\|g - g(t_0)\|_{A(I_m)} \to 0$ when $\{I_m\}$ are neighbourhoods of t_0 which shrink to t_0. This contradiction concludes the proof.

BIBLIOGRAPHY

1. N. K. Bari, *A treatise on trigonometric series*, Fizmatgiz, Moscow, 1961; English transl., Macmillan, New York; Pergamon Press, Oxford, 1964. MR 23 #A3411; 30 #1347.

2. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR 40 #1734.

3. N. Wiener, *The Fourier integral and certain of its applications*, Cambridge Univ. Press, Cambridge, 1933; reprint, Dover, New York, 1959. MR 20 #6634.

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