

ON A THEOREM OF MENCHOFF

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ABSTRACT. Construction of a continuous real-valued function which disagrees almost everywhere with any absolutely convergent Fourier series.

A classical theorem of Menchoff states that if f is a finite measurable function on the circle T and if $\epsilon > 0$, there exist functions g on T with uniformly convergent Fourier series and such that $f(t) = g(t)$ except possibly on a set of measure less than ϵ . The question has been raised (e.g. [1, problem 7, p. 527]) whether the theorem can be strengthened to yield, for any f and ϵ as above, an *absolutely convergent* Fourier series whose sum $g(t)$ agrees with $f(t)$ except for a set of t of measure less than ϵ . Our purpose in this note is to show that this cannot be done; in fact we prove more:

Theorem. *There exists a continuous real-valued function f on T such that if g is the sum of an absolutely convergent Fourier series then $\{t; f(t) = g(t)\}$ has measure zero.*

One can sharpen the problem by imposing on f a modulus of continuity; in fact one may conjecture that there exists an $f \in \text{Lip } 1/2$ which differs from any absolutely convergent Fourier series almost everywhere. The construction we give below yields a function whose modulus of continuity is bounded by $(-\log h)^{-1}$.

We denote by $A(T)$ the Banach algebra of absolutely convergent Fourier series. For a closed set $E \subset T$, we denote by $A(E)$ the Banach algebra of restrictions to E of the elements of $A(T)$, the norm being the standard (quotient) norm.

The basic tool is the so-called Rudin-Shapiro sequence $\{\epsilon_n\}$, $\epsilon_n = \pm 1$, $n = 0, 1, 2, \dots$, which has the property that for all positive integers N, M

$$(1) \quad \left\| \sum_{N+1}^{N+M} \epsilon_n e^{int} \right\|_{C(T)} \leq 5\sqrt{M}$$

(cf. [2, pp. 33–34]).

Define for $x \geq 0$,

$$(2) \quad \Phi(x) = \begin{cases} \epsilon_n, & n \leq x \leq n + \frac{1}{2}, \\ \text{linear for } n + \frac{1}{2} \leq x \leq n + 1, \end{cases}$$

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$$(3) \quad \phi_k(t) = \Phi(8^k t / 2\pi), \quad 0 \leq t \leq 2\pi.$$

One checks in the construction of $\{\epsilon_n\}$ that $\epsilon_0 = \epsilon_{8^k} = 1$ for all k and, thus, $\phi_k(0) = \phi_k(2\pi) = 1$, and we continue ϕ_k by periodicity or consider it as a function on $T = R/2\pi Z$.

We now write $k_m = m!$ and $a_m = k_m^{-1}$, notice that

$$(4) \quad a_m^{-1} 2^{3k_m - 1 - k_m} = o(1), \quad a_m^{-1} a_{m+1} = o(1), \quad a_m k_m = 1,$$

and claim that

$$(5) \quad f = \sum a_m \phi_{k_m}$$

satisfies the condition imposed in the theorem. The proof uses the following simple

Lemma. *Let k be a positive integer, l and n positive integers such that $l + n \leq 8^k$. Assume that $g \in C(T)$ and for some real-valued c and some α , $0 \leq \alpha \leq 1/2$, we have*

$$(6) \quad |g(t) - \phi_k(t) - c| < 1/10$$

on 90% of the points of E where $E = \{2\pi(\alpha + j)8^{-k}\}_{j=l}^{l+n}$. Then $\|g\|_{A(E)} > \pi^{-1} \log n - 4$.

Proof. Denote $\mu = \sum_{j=l}^{l+n} \epsilon_j \delta_{2\pi(\alpha+j)8^{-k}}$, that is, the measure carried by E whose mass at $2\pi(\alpha + j)8^{-k}$ is equal to ϵ_j . By (1) we have $\|\hat{\mu}\|_\infty \leq 5\sqrt{n}$.

Select a real number c_1 so that $c_1 = c \pmod{4}$ and $|c_1| \leq 2$. By (6), we have

$$\int \sin \frac{\pi}{2} (\operatorname{Re} g - c_1) d\mu > \frac{7}{10} n.$$

Thus,

$$\frac{7}{10} n \leq \left\| \sin \frac{\pi}{2} (\operatorname{Re} g - c_1) \right\|_{A(E)} \|\hat{\mu}\|_\infty \leq 5\sqrt{n} \left\| \sin \frac{\pi}{2} (\operatorname{Re} g - c_1) \right\|_{A(E)},$$

and

$$\frac{7}{50} \sqrt{n} \leq \left\| \sin \frac{\pi}{2} (\operatorname{Re} g - c_1) \right\|_{A(E)} \Rightarrow \|g\|_{A(E)} \geq \frac{2}{\pi} \log \frac{7\sqrt{n}}{50} - |c_1|,$$

that is

$$\|g\|_{A(E)} \geq \pi^{-1} \log n - (2 + 2\pi^{-1} \log(50/7));$$

and the lemma follows.

We now turn to the proof that the function f defined by (5) satisfies the condition stated in the theorem.

Let $g \in A(T)$ and assume that $G = \{t; g(t) = f(t)\}$ has positive measure.

Let t_0 be a point of density of G . For all sufficiently large m , the relative

measure of G in $I_m = (t_0 - 2\pi 2^{-k_m}, t_0 + 2\pi 2^{-k_m})$ is greater than 99%.

It is clear that for an appropriate $\alpha_m, 0 \leq \alpha_m \leq 1/2$, G contains at least 98% of the points of the form $2\pi(\alpha_m + j)8^{-k_m}$ which are contained in I_m . We denote the set of all these points by E_m . E_m is an arithmetical progression of length $2 \cdot 4^{k_m}$.

We now write

$$f = \sum_1^{m-1} a_j \phi_{k_j} + a_m \phi_{k_m} + \sum_{m+1}^{\infty} a_j \phi_{k_j}$$

and notice that the first sum differs from a constant on I_m by at most $2^{2-k_m} \sum_1^{m-1} a_j 8^{kj} \ll 2^{3k_m-1-k_m}$ (the half length of I_m being $2\pi 2^{-k_m}$ and the maximum slope of the sum is bounded by $(2\pi)^{-1} 4 \sum_1^{m-1} a_j 8^{kj}$).

On the other hand, $\sum_{m+1}^{\infty} a_j \phi_{k_j}$ is uniformly bounded by $2a_{m+1}$. Thus

$$a_m^{-1}(g - g(t_0)) = \phi_{k_m} + \text{const} + o(1)$$

on more than 90% of the points of E_m and it follows by the lemma that

$$\|a_m^{-1}(g - g(t_0))\|_{A(E_m)} \geq \pi^{-1} k_m$$

and so

$$\|g - g(t_0)\|_{A(I_m)} > \pi^{-1} a_m k_m = \pi^{-1}.$$

On the other hand, a classical theorem of Wiener [3, p. 88] implies that $\|g - g(t_0)\|_{A(I_m)} \rightarrow 0$ when $\{I_m\}$ are neighbourhoods of t_0 which shrink to t_0 . This contradiction concludes the proof.

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