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PROPERTIES OF WEAK $\overline{\theta}$-REFINABLE SPACES

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ABSTRACT. A space X is called *weak* $\overline{\theta}$*-refinable* if every open cover of X has a refinement $\bigcup_{i=1}^{\infty}\mathcal{G}_i$ satisfying (1) $\mathcal{G}_i = \{G_\alpha : \alpha \in A_i\}$ is an open collection for each i, (2) each $x \in X$ has finite positive order with respect to some $\mathcal{G}_i$, (3) the open cover $\{G_i = \bigcup[G_\alpha: \alpha \in A_i]\}_{i=1}^{\infty}$ is point finite.

In this paper the author shows that the above property lies between the properties of θ-refinable and weak θ-refinable. The main result is the fact that if X is countably metacompact and satisfies property (δ), every weak $\overline{\theta}$-cover of X has a countable subcover. Results concerning paracompactness, metacompactness and the star-finite property are also obtained.

Introduction. In [8] J. M. Worrell and H. H. Wicke announced the following result without proof.

Theorem 1.1. *Let X be a countably compact space. If $\mathcal{G} = \bigcup_{i=1}^{\infty}\mathcal{G}_i$ is an open cover of X such that for each $x \in X$, there exists an integer $n(x)$ such that $0 < \operatorname{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_0$, then $\mathcal{G}$ has a countable subcover.*

This result answers in the affirmative, the following two questions raised by H. R. Bennett [3], [4] and D. J. Lutzer [4].

(1) Are countably compact, weak θ-refinable spaces compact?

(2) Are countably compact, quasi-developable spaces metrizable?

In this paper we obtain a somewhat different result which also answers these questions. That is, if the property of the space is weakened while the type of open cover is strengthened to a weak $\overline{\theta}$-cover, then we can obtain a result analogous to Theorem 1.1 above. In §2 of this paper we introduce the notion of a weak $\overline{\theta}$-refinable space and show that this class of spaces lies between the class of θ-refinable spaces and the class of weak θ-refinable spaces. In §3 we obtain the main result, which also answers the above questions. Other properties of weak $\overline{\theta}$-refinable spaces are discussed in §4 and some open questions are included.

2. Weak $\overline{\theta}$-refinable spaces.

Definition 2.1. A space X is called *weak* $\overline{\theta}$*-refinable* if every open cover of X has an open refinement $\bigcup_{i=1}^{\infty}\mathcal{G}_i$ satisfying:

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(1) $\mathcal{G}_i = \{G_\alpha : \alpha \in A_i\}$ is an open collection for each i.

(2) For each $x \in X$, some $\mathcal{G}_i$ has finite, positive order at x.

(3) The open cover $\{G_i = \bigcup_{\alpha \in A_i} G_\alpha\}_{i=1}^{\infty}$ is point finite.

It is clear from the definitions that weak $\bar{\bar{\theta}}$-refinable spaces are weak θ-refinable, in the sense of Bennett and Lutzer [4].

Theorem 2.2. *Let X be a θ-refinable space. Then X is weak $\bar{\bar{\theta}}$-refinable.*

Proof. Suppose that X is θ-refinable and let $\mathcal{G}$ be an open cover of X. Then $\mathcal{G}$ has an open refinement $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying:

(1) $\mathcal{G}_i = \{G_\alpha : \alpha \in A_i\}$ is an open cover of X for each i.

(2) For each $x \in X$, some $\mathcal{G}_i$ has finite, positive order at x.

Now for each i and j, let

$$F(i,\ j) = \{x : \operatorname{ord}(x,\ \mathcal{G}_i) \leq j\}$$

so that $F(i, j)$ is closed in X. For each n define

$$\mathcal{G}_n^* = \left\{ G_\alpha^* = G_\alpha - \bigcup_{k=1}^{n-1} F(k,\ n-k) : \alpha \in A_n \right\}.$$

Clearly $\bigcup_{i=1}^{\infty} \mathcal{G}_i^*$ satisfies (1) and (2) in Definition 2.1 above. To see (3), let $x \in X$ and choose l to be the first integer such that $0 < \operatorname{ord}(x, \mathcal{G}_l^*) = m < \aleph_0$. Then $x \in F(l, m)$ and for $n > m + l$, $F(l, m) \subseteq F(l, n - l)$. Therefore x belongs to no member of $\mathcal{G}_n^*$ for $n > m + l$.

Example 2.3. In Example 1 of [4], Bennett and Lutzer showed that the space (R, τ) was weak θ-refinable but not θ-refinable. It is easy to show that this space is weak $\bar{\bar{\theta}}$-refinable. Therefore the property of weak $\bar{\bar{\theta}}$-refinable is strictly weaker than the property of θ-refinable.

3. The main result.

Definition 3.1. A space X has property (δ) if discrete collections in X are countable.

In [2] C. E. Aull defines a *$\delta\theta$-cover* as a famliy $\mathcal{O} = \bigcup_{i=1}^{\infty} \mathcal{O}_i$ of open sets such that $\mathcal{O}_i$ covers X and each $x \in X$ has countable order with respect to some $\mathcal{O}_i$. The following result, which generalizes a theorem of Aquaro [1], was also obtained in [2].

Theorem 3.2. *If X has property (δ), then every $\delta\theta$-cover has a countable subcover.*

We now prove an analogous result.

Theorem 3.3. *Let X be weak $\bar{\bar{\theta}}$-refinable and have property (δ). Then X is Lindelöf.*

Proof. Let $\mathfrak{U}$ be an open cover of X. Since X is weak $\bar{\bar{\theta}}$-refinable,

$\mathcal{G} > \bigcup_{i=1}^{\infty}\mathcal{G}_i$ where $\mathcal{G}_i = \{G_\alpha\colon \alpha \in A_i\}$ is an open collection such that

(1) $\{G_i\}_{i=1}^{\infty}$ is point finite, where $G_i = \bigcup_{\alpha \in A_i} G_\alpha$, and

(2) For each $x \in X$, there exists $n(x)$ such that $0 < \operatorname{ord}(x, \mathcal{G}_{n(x)}) < \aleph_0$.

Let $\mathcal{G}^* = \{G_1, G_2, \dots\}$. We construct for each k and j, a countable subcollection $\mathcal{O}(k, j)$ of $\mathcal{G}$ with the property that, if $0 < \operatorname{ord}(x, \mathcal{G}_i) < \aleph_0$ for some i, then x belongs to some member of $\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\mathcal{O}(k, j)$.

For each i and m define

$$\mathcal{B}(i, m) = \{B\colon B \subseteq A_i, |B| = m\} \quad \text{and} \quad S(i, m) = \{x\colon \operatorname{ord}(x, \mathcal{G}_i) \le m\}.$$

Step I. Let $k = 1$. We construct $\mathcal{O}(1, 1)$. Define $H_1 = \{x\colon \operatorname{ord}(x, \mathcal{G}^*) \le 1\}$, and for each i let $F(1, i, B, 1) = [G_i \cap H_1 \cap S(i, 1)] \cap [\bigcap_{\alpha \in B} G_\alpha]$ for each $B \in \mathcal{B}(i, 1)$.

We assert that $\mathcal{F}(1, i, 1) = \{F(1, i, B, 1)\colon B \in \mathcal{B}(i, 1)\}$ is discrete in X for each i. Let $x \in X$.

(1) If $x \notin G_i$, then $x \in G_l$ for $l \neq i$ and $G_l \cap [G_i \cap H_1] = \emptyset$.

(2) Let $x \in G_i$. If $\operatorname{ord}(x, \mathcal{G}_i) > 1$, there exist distinct α and $\beta \in A_i$ such that $x \in G_\alpha \cap G_\beta$. But $(G_\alpha \cap G_\beta) \cap S(i, 1) = \emptyset$. If $\operatorname{ord}(x, \mathcal{G}_i) = 1$, then x belongs to only one member of $\mathcal{G}_i$, say G_α. But G_α intersects only $F(1, i, B, 1)$ where $B = \{\alpha\}$.

Since X has property (δ), $\mathcal{F}(1, 1) = \bigcup_{i=1}^{\infty}\mathcal{F}(1, i, 1)$ is countable so that there exists a countable subfamily $\mathcal{O}(1, 1)$ of $\mathcal{G}$ such that each member of $\mathcal{F}(1, 1)$ is contained in some member of $\mathcal{O}(1, 1)$. Clearly $\mathcal{O}(1, 1)$ has the property that if $\operatorname{ord}(x, \mathcal{G}^*) = 1$ and $\operatorname{ord}(x, \mathcal{G}_i) = 1$ for some i, then x belongs to some member of $\mathcal{O}(1, 1)$.

Step II. We now construct $\mathcal{O}(1, j)$ by induction on j.

Suppose that $\mathcal{O}(1, j)$ has been constructed for $j \le m$ satisfying the property that if $\operatorname{ord}(x, \mathcal{G}^*) = 1$ and $\operatorname{ord}(x, \mathcal{G}_i) \le m$ for some i, then x belongs to some member of $\bigcup_{j=1}^{m}\mathcal{O}(1, j)$. Let $V(1, m) = \bigcup\{V\colon V \in \bigcup_{j=1}^{m}\mathcal{O}(1, j)\}$.

For each i and $B \in \mathcal{B}(i, m+1)$ define

$$F(1, i, B, m+1) = [G_i \cap H_1 \cap S(i, m+1)] \cap [X - V(1, m)] \cap \left[\bigcap_{\alpha \in B} G_\alpha\right].$$

We assert that $\mathcal{F}(1, i, m+1) = \{F(1, i, B, m+1)\colon B \in \mathcal{B}(i, m+1)\}$ is discrete for each i. Let i be fixed.

As in Step I if $x \notin G_i$, then x has a neighborhood which misses $H_1 \cap G_i$. Let $x \in G_i$.

(1) If $\operatorname{ord}(x, \mathcal{G}_i) < m+1$, then $x \in V(1, m)$ which misses all members of $\mathcal{F}(1, i, m+1)$.

(2) If $\operatorname{ord}(x, \mathcal{G}_i) > m+1$, then x belongs to at least $m+2$ members of $\mathcal{G}_i$, say $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_{m+2}}$. Then $\bigcap_{l=1}^{m+2} G_{\alpha_l}$ is a neighborhood of x which misses $S(i, m+1)$.

(3) If $\operatorname{ord}(x, \mathcal{G}_i) = m + 1$, then x belongs to exactly $m + 1$ members of $\mathcal{G}_i$, say $G_{\alpha_1}, \ldots, G_{\alpha_{m+1}}$. Thus $\bigcap_{l=1}^{m+1} G_{\alpha_l}$ intersects only $F(1, i, B, m + 1)$ where $B = \{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}\}$.

Since X has property (δ) there exists a countable subfamily $\mathcal{O}(1, m + 1)$ of $\mathcal{G}$ such that each member of $\mathcal{F}(1, m + 1) = \bigcup_{i=1}^{\infty} \mathcal{F}(1, i, m + 1)$ is contained in some member of $\mathcal{O}(1, m + 1)$. Therefore, by construction, if $\operatorname{ord}(x, \mathcal{G}^*) = 1$ and $0 < \operatorname{ord}(x, \mathcal{G}_i) \leq m + 1$ for some i, then x belongs to some member of $\bigcup_{j=1}^{m+1} \mathcal{O}(1, j)$. The induction is now complete on j.

Step III. We now assert that for each k, there exists a sequence of countable open subfamilies $\{\mathcal{O}(k, j)\colon j = 1, 2, \ldots\}$ satisfying the property that if $\operatorname{ord}(x, \mathcal{G}^*) \leq n$ and $0 < \operatorname{ord}(x, \mathcal{G}_i) < \aleph_0$ for some i, then x belongs to some member of $\bigcup_{k=1}^{n} \bigcup_{j=1}^{\infty} \mathcal{O}(k, j)$.

The proof is by induction on k. The case for $k = 1$ has been proved in Steps I and II above.

Assume that the subfamilies with the above property exist for all $l < k$. We show how to construct $\{\mathcal{O}(k, j)\colon j = 1, \ldots\}$.

(A) Let $j = 1$. Define $V^* = \bigcup\{V\colon V \in \bigcup_{l=1}^{k-1} \bigcup_{j=1}^{\infty} \mathcal{O}(l, j)\}$ and $H_k = \{x\colon \operatorname{ord}(x, \mathcal{G}^*) \leq k\}$. For each i, let

$$F(k, i, B, 1) = [G_i \cap H_k \cap S(i, 1)] \cap [X - V^*] \cap \left[\bigcap_{\alpha \in B} G_\alpha\right]$$

for each $B \in \mathcal{B}(i, 1)$. As before, $\mathcal{F}(k, i, 1) = \{F(k, i, B, 1)\colon B \in \mathcal{B}(i, 1)\}$ is discrete for each i.

Indeed, let $x \in X$ and i fixed.

(1) If $\operatorname{ord}(x, \mathcal{G}^*) > k$, then x has a neighborhood which misses H_k.

(2) If $\operatorname{ord}(x, \mathcal{G}^*) < k$, then $x \in V^*$ since $0 < \operatorname{ord}(x, \mathcal{G}_l) < \aleph_0$ for some l.

(3) Suppose $\operatorname{ord}(x, \mathcal{G}^*) = k$.

Case 1. If $x \notin G_i$, then x belongs to k other members of $\mathcal{G}^*$, say $G_{j_1}, G_{j_2}, \ldots, G_{j_k}$. Hence $[\bigcap_{l=1}^{k} G_{j_l}] \cap [G_i \cap H_k] = \varnothing$.

Case 2. Let $x \in G_i$. If $\operatorname{ord}(x, \mathcal{G}_i) > 1$, then x has a neighborhood which misses $S(i, 1)$. If $\operatorname{ord}(x, \mathcal{G}_i) = 1$, then x has a neighborhood which intersects exactly one member of $\mathcal{F}(k, i, 1)$ as in Step I above.

Therefore there exists a countable subfamliy $\mathcal{O}(k, 1)$ of $\mathcal{G}$ satisfying the required property. We construct $\mathcal{O}(k, j)$ by induction on j.

(B) Assume that $\mathcal{O}(k, j)$ has been constructed for $j \leq m$ with the property that if $\operatorname{ord}(x, \mathcal{G}^*) < k$ or $\operatorname{ord}(x, \mathcal{G}^*) = k$ and $0 < \operatorname{ord}(x, \mathcal{G}_i) \leq m$ for some i, then x belongs to some member of $V^{**} = [\bigcup_{j=1}^{m} \mathcal{O}(k, j)] \cup V^*$.

We now construct $\mathcal{O}(k, m + 1)$. For fixed i, define

$$F(k, i, B, m + 1) = [G_i \cap H_k \cap S(i, m + 1)] \cap [X - V^{**}] \cap \left[\bigcap_{\alpha \in B} G_\alpha\right]$$

for each $B \in \mathcal{B}(i, m + 1)$. Again, $\mathcal{F}(k, i, m + 1) = \{F(k, i, B, m + 1)\colon B \in$

$\mathcal{B}(i, m+1)\}$ is discrete for each i. The argument is essentially the same as that in Step II above, and hence is omitted.

Therefore there exists a countable subfamily $\mathcal{O}(k, m+1)$ of $\mathcal{G}$ such that each member of $\mathcal{F}(k, m+1) = \bigcup_{i=1}^{\infty}\mathcal{F}(k, i, m+1)$ is contained in some member of $\mathcal{O}(k, m+1)$.

Since the induction on j is complete, the induction on k is also complete. It is easy to show that if $\operatorname{ord}(x, \mathcal{G}^*) \leq k$, then x belongs to some member of $\bigcup_{l=1}^{k}\bigcup_{j=1}^{\infty}\mathcal{O}(l, j)$. Thus $\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\mathcal{O}(k, j)$ is the desired subcover of X.

Lemma 3.4. *Let X be a countably metacompact space and $\mathcal{G} = \{G_i\}_{i=1}^{\infty}$ a countable open cover of X. Then for each integer $k > 0$, there exists a point finite open refinement $\mathcal{O}(k) = \{V(k, j)\}_{j=1}^{\infty}$ such that $V(k, k) = G_k$.*

Proof. Let $k > 0$ be given. Define

$$W(k, j) = G_j \text{ for } j \leq k,$$

$$= G_j - \bigcup_{i<j} G_i \text{ for } j > k.$$

Note that $\{W(k, j)\}_{j=1}^{\infty}$ is a locally finite cover of X.

Since X is countably metacompact and, hence, almost $\aleph_0$-expandable by Theorem 1.6 of [7], there exists a point finite collection $\mathcal{O}(k) = \{V(k, j)\}_{j=1}^{\infty}$ such that $W(k, j) \subseteq V(k, j) \subseteq G_j$ for each j. Clearly $\mathcal{O}(k)$ can easily be modified to satisfy the above property.

Theorem 3.5. *Let X be a countably metacompact space satisfying property (δ). Then every weak θ-cover has a countable subcover.*

Proof. Let $\mathcal{G} = \bigcup_{i=1}^{\infty}\mathcal{G}_i$ be a weak θ-cover of X. Define $G_i^* = \bigcup\{G: G \in \mathcal{G}_i\}$ for each i, so that $\{G_i^*\}_{i=1}^{\infty}$ is a countable open cover of X. For each $k > 0$, $\{G_i^*\}_{i=1}^{\infty}$ has a point finite open refinement $\mathcal{W}(k) = \{W(k, j)\}_{j=1}^{\infty}$ such that $W(k, k) = G_k^*$ by Lemma 3.5. Now for each $k > 0$, define $\mathcal{U}_k = \bigcup_{i=1}^{\infty}[\mathcal{G}_i \cap W(k, i)]$. By a modification of the same technique used in the proof of Theorem 3.3 above, each $\mathcal{U}_k$ has a countable subcollection $\mathcal{O}_k$ such that $\mathcal{O} = \bigcup_{k=1}^{\infty}\mathcal{O}_k$ covers X. Therefore $\mathcal{G}$ has a countable subcover.

Corollary 3.6. (1) *Every countably metacompact, weak θ-refinable space satisfying property (δ) is Lindelöf.*

(2) *Every countably compact, weak θ-refinable space is compact.*

(3) *Every countably compact, quasi-developable space is metrizable.*

(4) *Every T_2 quasi-developable, M-space is metrizable.*

4. **Applications and questions.** In [7] L. Krajewski and the author studied the properties of various types of expandable spaces. One result is the following.

Theorem 4.1. *Let X be a regular, discretely H.C. expandable space. Then the following are equivalent.*

(1) *X is paracompact.*

(2) *X is subparacompact.*

(3) *X is metacompact.*

(4) *X is θ-refinable.*

We now can generalize this to the following.

Theorem 4.2. *Let X be a regular, discretely H.C. expandable space. Then X is paracompact iff X is weak $\overline{\theta}$-refinable.*

Proof. The proof is essentially the same as that for Theorem 3.3 above, where at each stage the discrete collections obtained, which need not be countable, are expanded to hereditarily closure preserving open collections that refine the original cover. Thus $\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\mathcal{O}(k, j)$ is a σ-closure preserving open refinement; and hence X is paracompact by Theorem 2 of [5].

In a similar fashion we can also obtain the next two results.

Theorem 4.3. *Let X be an almost expandable space. Then X is metacompact if X is weak $\overline{\theta}$-refinable.*

Proof. As in Theorem 4.2 above every open cover of X has a σ-point finite open refinement. Since almost expandable spaces are countably metacompact, X is metacompact.

Theorem 4.4. *Every almost expandable, weak θ-refinable space is metacompact.*

Theorem 4.5. *Let X be a regular, weak $\overline{\theta}$-refinable space with property (δ). Then X has the star finite property.*

Proof. X is Lindelöf and hence normal so that every open cover has a countable cozero refinement. Thus X has the star finite property by a result of Morita [6].

Several open questions still remain.

(1) Are countably metacompact, weak θ-refinable spaces weak $\overline{\theta}$-refinable?

(2) Are almost discretely expandable, weak $\overline{\theta}$-refinable spaces metacompact?

(3) Are quasi-developable spaces weak $\overline{\theta}$-refinable?

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