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CONJUGATE ALGEBRAIC INTEGERS IN AN INTERVAL

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ABSTRACT. The following conjecture of R. M. Robinson is proved. If Δ is a real interval of length greater than 4, then for any sufficiently large n there exists an irreducible monic polynomial of degree n with integer coefficients all of whose zeros lie in Δ.

The purpose of this note is to prove the following result.

Theorem. *Let Δ be any real interval of length greater than 4. Then for any sufficiently large n there exists an irreducible monic polynomial of degree n with integer coefficients all of whose zeros lie in Δ.*

This settles a conjecture proposed by R. M. Robinson [2] and repeated in [1, Problem 2, p. 467]. Robinson proved the result under the further restriction that n is divisible by a suitably chosen integer m depending on Δ.

Proof. Without loss of generality, we may assume that Δ is a closed interval $[c-2\lambda, c+2\lambda]$, where c and λ are rational and $\lambda > 1$. For $n \geq 0$, let $T_n(x)$ denote the Chebyshev polynomial of degree n, defined by $T_n(2\cos\theta) = 2\cos n\theta$. Then $T_n'(2\cos\theta) = n\sin n\theta/\sin\theta$, whence $|T_n'(x)| \leq n^2$ for $|x| \leq 2$. Put $P_n(x) = \lambda^n T_n((x-c)/\lambda)$. Then $P_0(x) = 2$ and for $n > 0$, $P_n(x)$ is a monic polynomial of degree n with rational coefficients having all its zeros within Δ. The function $P_n(x)$ oscillates n times between the bounds $\pm 2\lambda^n$ in the interval Δ. Further

$$(1) \qquad |P_n'(x)| = \lambda^{n-1}|T_n'((x-c)/\lambda)| \leq n^2\lambda^{n-1} \qquad (n = 0, 1, \ldots; x \in \Delta).$$

Write $P_n(x) = x^n + \sum_{k=1}^n a_{nk}x^{n-k}$. As indicated by Robinson [2], the coefficients a_{nk} are polynomials in n of degree k with rational coefficients containing n as a factor. Hence we can write

$$a_{nk} = (\alpha_{k0}n^k + \cdots + \alpha_{k,k-1}n)/\beta_k \qquad (1 \leq k \leq n),$$

for some integers α_{ki}, β_k. Choose a natural number l so large that $\lambda^l(\lambda - 1) \geq 2$, and then take m to be the least common multiple of $\beta_1, \ldots, \beta_l$. For any integer n define $r(n)$ by $r(n) \equiv n \bmod m$, $0 \leq r(n) < m$. Let h be an even integer such that $c - \lambda \leq h \leq c + \lambda$. Let $n_0 = n_0(c, \lambda) \geq l$ be a suitable sufficiently large number.

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The method of Robinson depends on taking $n \equiv 0 \bmod m$, so that the coefficients $a_{n1}, \ldots, a_{nl}$ in $P_n(x)$ are integers. One can then change $P_n(x)$ slightly to obtain a monic irreducible polynomial $Q_n(x)$ with integer coefficients having a similar behaviour as $P_n(x)$ in the interval Δ. We shall take these polynomials as a starting point and then use induction from jm to $jm+m-1$ to construct a polynomial $Q_n(x)$ for each $n \geq n_0$ having the following properties. $Q_n(x)$ is a monic polynomial of degree n with integer coefficients and an Eisenstein polynomial with respect to the prime 2. All its zeros, $\xi_{n1} < \xi_{n2} < \cdots < \xi_{nn}$, are real and contained in Δ. Define $\omega_{n0} = c - 2\lambda$, $\omega_{nn} = c + 2\lambda$, and for $1 \leq i \leq n-1$, ω_{ni} by $\xi_{ni} < \omega_{ni} < \xi_{n,i+1}$, $Q_n'(\omega_{ni}) = 0$. Finally, $t(n)$ is determined by $\omega_{n,t(n)} \leq h < \omega_{n,t(n)+1}$. Then the polynomial $Q_n(x)$ will further satisfy

$$(2.n) \qquad |Q_n(x)| < 16^{r(n)+1}\lambda^n \qquad (x \in \Delta),$$

$$(3.n) \qquad |Q_n'(x)| < 16^{r(n)+1}n^2\lambda^{n-1} \qquad (x \in \Delta),$$

$$(4.n) \qquad (-1)^{n-i}Q_n(\omega_{ni}) > (16n)^{-4^{r(n)}}\lambda^n \qquad (i = 0, 1, \ldots, n),$$

$$(5.n) \qquad 1 \leq t(n) \leq n-2.$$

For $n \equiv 0 \bmod m$ we use the original construction of Robinson, viz.

$$Q_n(x) = P_n(x) + \sum_{k=l+1}^{n} b_k P_{n-k}(x),$$

where the b_k are suitably chosen rational numbers with $-1 \leq b_k < 1$. As in [2], it follows easily from the choice of l that $(2.n)$ and $(4.n)$ hold. Similarly, $(3.n)$ follows from (1). From the behaviour of $Q_n(x)$ it is easy to deduce that $(5.n)$ is true, because h does not lie near the endpoints of Δ.

Supposing that we have constructed $Q_n(x)$ for some $n \geq n_0$, $n \not\equiv -1 \bmod m$, we take

$$Q_{n+1}(x) = (x-h)(Q_n(x) + 2\epsilon[s\lambda^n/4]) + 2,$$

where $\epsilon = (-1)^{n-t(n)}$ or $\epsilon = (-1)^{n-t(n)-1}$ according as $\omega_{n,t(n)} \leq h < \xi_{n,t(n)+1}$ or $\xi_{n,t(n)+1} < h < \omega_{n,t(n)+1}$, $s = (16n)^{-4^{r(n)}}$, and $[z]$ denotes the integral part of z. It is clear that $Q_{n+1}(x)$ is monic of degree $n+1$ with integer coefficients, and is an Eisenstein polynomial with respect to 2. Simple estimates show that $(2.n+1)$ and $(3.n+1)$ are true. From $(3.n)$ and $(4.n)$ we obtain

$$(6) \qquad \min_{i=1,\ldots,n} \{\omega_{ni} - \xi_{ni}, \xi_{ni} - \omega_{n,i-1}\} > \delta,$$

where $\delta = 16^{-r(n)-1}n^{-2}s\lambda$.

Consider the case $\omega_{n,t(n)} \leq h < \xi_{n,t(n)+1}$ so that $\epsilon = \operatorname{sgn} Q_n(\omega_{n,t(n)})$. In the other case the argument is similar. From $(4.n)$ and (6) we have

$$(7) \qquad \epsilon_i Q_{n+1}(\omega_{ni}) > 16^{-r(n)-2}n^{-2}s^2\lambda^{n+1} \qquad (i = 0, \ldots, n;\ i \neq t(n)),$$

where $\epsilon_i = (-1)^{n+1-i}$ for $0 \leq i \leq t(n) - 1$, and $\epsilon_i = (-1)^{n-i}$ for $t(n) + 1 \leq i \leq n$. Using (3.n) we obtain

$$\begin{aligned}
&\min\{-\epsilon Q_{n+1}(\omega_{n,t(n)} - \tfrac{1}{4}\delta), \epsilon Q_{n+1}(\xi_{n,t(n)+1} + \tfrac{1}{4}\delta)\} \\
&\quad > \tfrac{1}{4}\delta(\tfrac{1}{2}s\lambda^n - \tfrac{1}{4}\delta 16^{r(n)+1} n^2 \lambda^{n-1} - 2) - 2 \\
(8) \qquad &\quad = 16^{-r(n)-2} n^{-2} s^2 \lambda^{n+1} - \tfrac{1}{2}\delta - 2 \\
&\quad > (16(n+1))^{-4^{r(n)+1}} \lambda^{n+1}.
\end{aligned}$$

By (7) and (8), it is now easy to see that the polynomial $Q_{n+1}(x)$ has the required behaviour, and that (4.$n+1$) holds. Clearly also (5.$n+1$) is true. Hence the induction is complete, and the Theorem follows.

REFERENCES

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