PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

SOME COMMUTATIVITY RESULTS FOR RINGS WITH TWO-VARIABLE CONSTRAINTS

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ABSTRACT. It is proved that an associative ring R has nil commutator ideal if for each $x, y \in R$, there is a polynomial $p(X) \in X\mathbf{Z}[X]$ for which $xy - yp(x)$ is central. Two restrictions on the $p(X)$ which guarantee commutativity are established.

Let $\mathcal{P}$ denote the set of those polynomials in two noncommuting indeterminates which have integer coefficients and constant term zero. We consider associative rings R with the property that for each ordered pair (x, y) of elements of R, there exists a polynomial $p(X, Y) \in \mathcal{P}$, depending on (x, y), for which

$$xy - p(x, y) \in Z, \tag{1}$$

where Z denotes the center of R.

Putcha and Yaqub [6] have shown that if each $p(X, Y)$ in (1) is a sum of terms each of degree at least two in both X and Y, then $R^2 \subseteq Z$, and hence, by a long-standing theorem of Herstein [4], R has nil commutator ideal. Unless the $p(X, Y)$ in (1) are restricted in some fashion, R may be badly noncommutative—indeed the ring of 2×2 matrices over $GF(2)$ satisfies a condition of type (1), obtained by linearizing the identity $x^2 = x^8$. However, less severe restrictions than those imposed by Putcha and Yaqub, while not implying that any power of R is central, will still yield the result that R has nil commutator ideal; and this note deals with one such condition, together with some special cases of it which actually yield commutativity.

Letting $X\mathbf{Z}[X]$ denote the ring of polynomials over the integers which have zero constant term, we state our major theorem as follows:

Theorem 1. *Let R be a ring such that for each ordered pair (x, y) of elements of R there exists a polynomial $p(X) \in X\mathbf{Z}[X]$, depending on (x, y), for which*

$$xy - yp(x) \in Z. \tag{2}$$

Then the commutator ideal $C(R)$ is nil and the nilpotent elements of R form an ideal.

1. Proof of Theorem 1.

Lemma 1. *Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of noncommuting indeterminates, its coef-*

Received by the editors June 4, 1974 and, in revised form, October 14, 1974.
AMS (MOS) subject classifications (1970). Primary 16A70.

ficients being integers with highest common factor 1. *If there exists no prime* p *for which the ring of* 2×2 *matrices over* $GF(p)$ *satisfies* $q(X) = 0$, *then* R *has nil commutator ideal and the nilpotent elements of* R *form an ideal.*

The proof of this lemma, which depends on a deep result of Amitsur on PI-rings, may be found in [2].

Lemma 2. *Let* R *be a ring satisfying the hypothesis of Theorem* 1 *and having no nonzero divisors of zero*; *and let* (x, y) *be an arbitrary ordered pair of elements of* R. *If* $p(X) \in X\mathbf{Z}[X]$ *is such that* $xy - yp(x) \in Z$, *then* $xy^2 = y^2x$ *or* $xy = yp(x)$.

Proof. Suppose that $xy^2 \neq y^2x$, and write

$$xy = yp(x) + z, \quad \text{where } z \in Z; \tag{3}$$

and let $p_1(X) \in X\mathbf{Z}[X]$ be such that

$$x^2y - yp_1(x^2) \in Z. \tag{4}$$

Repeated substitution of (3) in (4) yields $x(yp(x) + z) - yp_1(x^2) \in Z$, $(yp(x) + z)p(x) + xz - yp_1(x^2) \in Z$, and finally

$$y((p(x))^2 - p_1(x^2)) + z(x + p(x)) \in Z. \tag{5}$$

If $(p(x))^2 - p_1(x^2) \neq 0$, (5) implies that $xy = yx$, contrary to our supposition that $xy^2 \neq y^2x$; hence

$$(p(x))^2 - p_1(x^2) = 0 \quad \text{and} \quad z(p(x) + x) \in Z, \tag{6}$$

so that $z = 0$ or $p(x) + x \in Z$. But if $p(x) + x \in Z$, then (3) yields $xy - yp(x) = xy - y(p(x) + x) + yx \in Z$, implying that y commutes with $xy + yx$ and, hence, that y^2 commutes with x; therefore $z = 0$ and (3) now shows that $xy = yp(x)$.

Proof of Theorem 1. It will suffice to show that prime rings satisfying the hypothesis of Theorem 1 are commutative (see [2]). Accordingly, let R be such a prime ring; we first show that R has no nonzero divisors of zero. Suppose that $ab = 0$, $a \neq 0$, and r is an arbitrary element of R. There exists $q(X) \in X\mathbf{Z}[X]$ for which $b(ra) - (ra)q(b) \in Z$; and since $aq(b) = 0$, we have $b(ra) \in Z$ and thus $sa(bra) = 0 = (bra)sa$ for all $s \in R$. The primeness of R now implies that $bra = 0$ and hence that $b = 0$.

Assume that R is a *noncommutative* prime ring satisfying (2). The identity

$$(xy^2 - y^2x)(yx^2 - x^2y)(xy^2x - yx^2y) = 0 \tag{7}$$

is not satisfied by the ring of 2×2 matrices over any field $GF(p)$, as may be verified by substituting the matrices $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for x and y respectively; thus, by Lemma 1, R cannot satisfy (7), and there must exist elements a, b of R for which $ab^2 - b^2a$, $ba^2 - a^2b$, and $ab^2a - ba^2b$ are all nonzero.

If $p(X) \in X\mathbf{Z}[X]$ is such that $ab - bp(a) \in Z$, it follows from Lemma 2 that

(8) $$ab = bp(a).$$

Now let $s(X) \in X\mathbf{Z}[X]$ satisfy

(9) $$bp(a) - p(a)s(b) \in Z$$

and apply the result of Lemma 2 to the ordered pair $(b, p(a))$. If $(p(a))^2 b = b(p(a))^2$, it follows from (8) that $a^2 b = a(bp(a)) = b(p(a))^2 = (p(a))^2 b$, so that $a^2 = (p(a))^2$ and a^2 commutes with b, contrary to the choice of a and b. Therefore, by Lemma 2, $bp(a) = p(a)s(b)$, which combines with (8) to give

(10) $$ab = p(a)s(b).$$

Now it is immediate from Lemma 2 that R is an Ore domain and can be embedded in a division ring D. In D, (10) implies that $b(s(b))^{-1} = a^{-1}p(a)$ commutes with both a and b; and (8) written in the form $ab = baa^{-1}p(a)$ shows that ab and ba commute, contrary to the original choice of a and b. This contradiction completes the proof of Theorem 1.

2. **Two commutativity theorems.** In this section we single out two conditions of type (2) which imply commutativity.

Theorem 2. *Let R be a ring such that for every ordered pair (x, y) of elements of R, there exists an integer $n = n(x, y) \geq 1$ for which $xy = yx^n$. Then R is commutative.*

Lemma 3. *Any ring R satisfying the hypothesis of Theorem 2 has each of the following properties*:

(a) *Idempotents of R are central.*

(b) *R is a duo ring (i.e. one-sided ideals are two-sided); moreover $ab = 0$ implies $ba = 0$, so that there is no distinction between right and left zero divisors.*

(c) *Commutators in R are central.*

(d) *If $a, b \in R$ are such that $a(ab - ba) = b(ab - ba) = 0$, then $ab - ba = 0$; similarly, if $a(ab - ba)x = b(ab - ba)x = 0$ for some $x \in R$, then $(ab - ba)x = 0$.*

Proof. (a) If $x \in R$ and e is idempotent, there exist positive integers m, n such that $e(ex - exe) = (ex - exe)e^m$ and $e(xe - exe) = (xe - exe)e^n$; hence $ex - exe = xe - exe = 0$.

(b) Let I be a right ideal of R, $a \in I$ and $r \in R$; note that since $ra = ar^n$ for some $n \geq 1$, $ra \in I$. Thus all right ideals are two-sided, and a similar argument holds for left ideals.

Now let $ab = 0$. Since $ba = ab^n$ for some $n \geq 1$, $ba = 0$ as well.

(c) By Theorem 1 the commutator ideal is nil and, hence, contained in the Jacobson radical $J(R)$; therefore, it will suffice to show $J(R) \subseteq Z$. If we assume the existence of an element $a \in J(R)\setminus Z$, then there is an element $b \in R$ and integers $m, n > 1$ for which $ab = ba^m$ and $ba = ab^n \neq ab$. It follows that $ab = abb^{n-1}a^{m-1}$; and because $b^{n-1}a^{m-1} \in J(R)$, we now have $ab = 0$. Similarly, $ba = 0$ and we have a contradiction.

(d) Suppose $a(ab - ba) = b(ab - ba) = 0$; in view of (c), $a^2b = ba^2$ and $b^2a = ab^2$. Suppose $ab - ba \neq 0$ and let $m, n > 1$ be such that $ab = ba^m$ and $ba = ab^n$. Substituting each of these expressions into the other yields $ab = ab^n a^{m-1}$ and $ba = ba^m b^{n-1}$. If m and n are both even we thus get $ab = ba = a^m b^n$; on the other hand, if one of n, m is odd, we have

$$ab - ba = abb^{n-1}a^{m-1} - baa^{m-1}b^{n-1} = (ab - ba)a^{m-1}b^{n-1},$$

which is zero since $(ab - ba)a = 0$.

Finally, if $x \in R$ and A is the annihilator of x, we get the second statement of (d) by applying the preceding argument to the ring R/A.

Proof of Theorem 2. It will suffice to prove commutativity under the additional hypothesis that R is subdirectly irreducible, in which case (since R is a duo ring) the set of zero divisors is precisely the annihilator of the unique minimal ideal S [1, Lemma 3].

The initial step is to show that zero divisors in R are central. Accordingly, suppose a is a noncentral zero divisor which fails to commute with some element $b \in R$; and consider the case where b is also a zero divisor. Then by (d) of Lemma 2, we have one of $(ab - ba)a$ and $(ab - ba)b$ different from 0 and $(ab - ba)R$ is a nontrivial ideal; therefore if $0 \neq s \in S$, there exists an element $x \in R$ for which $s = (ab - ba)x$. But $0 = as = bs = a(ab - ba)x = b(ab - ba)x$, and from (d) of Lemma 2 we then get $(ab - ba)x = 0$, a contradiction. Now consider the case where b is not a zero divisor and let $m, n > 1$ be such that $ab = ba^m$ and $ba = ab^n$. Since ab is a zero divisor, ab and a commute, so that $a(ab - ba) = (ab - ba)a = 0$ and a^2 commutes with b. If m is odd, repeating some of the computation in Lemma 2(d) shows that $ab - ba = (ab - ba)a^{m-1}b^{n-1} = 0$; on the other hand, if m is even, $ab = a^m b$, $a^m = a$, and a^{m-1} is a nonzero idempotent. Recalling that any nonzero central idempotent of a subdirectly irreducible ring must be a multiplicative identity element, we get a contradiction of the fact that a was a zero divisor. Therefore zero divisors of R are central.

Now suppose that R is not commutative and $b \notin Z$. There then exist $a \in R$ not commuting with b and an integer $j > 1$ such that $ba = ab^j$. Since a cannot be a zero divisor and since $ab - ba = a(b - b^j)$ is a zero divisor (nilpotent, in fact), $b - b^j$ must be a zero divisor, hence central. We have now arrived at a contradiction of Herstein's well-known result that a ring R is commutative if for each $x \in R$, there is an integer $n(x) > 1$ for which $x - x^{n(x)} \in Z$; and our proof is complete.

Theorem 3. *Let R be a ring such that for every ordered pair (x, y) of* *elements of* R, *there is a polynomial* $p(X) \in X\mathbf{Z}[X]$ *such that* $xy = yxp(x)$. *Then* R *is commutative.*

Proof. Again applying the given condition to e, $ex - exe$, and $xe - exe$ shows that idempotents must be central. Also, since $x^2 = x^2p(x)$ for some $p(X) \in X\mathbf{Z}[X]$, R is periodic by a result of Chacron [3]; therefore, R is either nil or contains a nonzero idempotent.

Suppose now that R is subdirectly irreducible. If R contains a nonzero idempotent, then it must have an identity; thus, for each $x \in R$ we have $x = xp(x)$, and R is commutative by the major theorem of [5]. On the other hand, if R is nil we have

$$xy = yxp(x) = xyq(y)p(x) = yxp(x)q(y)p(x) = yxq(y)r(x)$$

for an appropriate element $r(X) \in X\mathbf{Z}[X]$. In particular, $xy = yxyz_1$ for some element $z_1 \in R$; and, continuing inductively, for each positive integer n we get an element $z_n \in R$ for which $xy = y^nxyz_n$, so that $xy = 0$ and R is a zero ring. Therefore, if R is subdirectly irreducible, it is commutative; and the proof of Theorem 3 is finished.

The hypothesis of Theorem 3 cannot be weakened to the condition that $xy - yxp(x) \in Z$, as we see by noting that there exist noncommutative rings satisfying the identity $x^2 = 0$. However, it may be of some interest (but not enough to justify including the proof) to note that rings satisfying the weaker hypothesis are polynomial-identity rings—satisfying the identity $[[x, y], z]^2[x, y] = 0$.

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