SOME COMMUTATIVITY RESULTS FOR RINGS WITH TWO-VARIABLE CONSTRAINTS

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Abstract. It is proved that an associative ring $R$ has nil commutator ideal if for each $x, y \in R$, there is a polynomial $p(X) \in XZ[X]$ for which $xy - yp(x)$ is central. Two restrictions on the $p(X)$ which guarantee commutativity are established.

Let $\mathcal{P}$ denote the set of those polynomials in two noncommuting indeterminates which have integer coefficients and constant term zero. We consider associative rings $R$ with the property that for each ordered pair $(x, y)$ of elements of $R$, there exists a polynomial $p(X, Y) \in \mathcal{P}$, depending on $(x, y)$, for which
\[(1) \quad xy - p(x, y) \in Z,\]
where $Z$ denotes the center of $R$.

Putcha and Yaqub [6] have shown that if each $p(X, Y)$ in (1) is a sum of terms each of degree at least two in both $X$ and $Y$, then $R \subseteq Z$, and hence, by a long-standing theorem of Herstein [4], $R$ has nil commutator ideal. Unless the $p(X, Y)$ in (1) are restricted in some fashion, $R$ may be badly noncommutative—indeed the ring of $2 \times 2$ matrices over $GF(2)$ satisfies a condition of type (1), obtained by linearizing the identity $x^2 = x^8$. However, less severe restrictions than those imposed by Putcha and Yaqub, while not implying that any power of $R$ is central, will still yield the result that $R$ has nil commutator ideal; and this note deals with one such condition, together with some special cases of it which actually yield commutativity.

Letting $XZ[X]$ denote the ring of polynomials over the integers which have zero constant term, we state our major theorem as follows:

Theorem 1. Let $R$ be a ring such that for each ordered pair $(x, y)$ of elements of $R$ there exists a polynomial $p(X) \in XZ[X]$, depending on $(x, y)$, for which
\[(2) \quad xy - yp(x) \in Z.\]
Then the commutator ideal $C(R)$ is nil and the nilpotent elements of $R$ form an ideal.

1. Proof of Theorem 1.

Lemma 1. Let $R$ be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of noncommuting indeterminates, its coe-
ficients being integers with highest common factor 1. If there exists no prime p for which the ring of 2 \times 2 matrices over GF(p) satisfies q(X) = 0, then R has nil commutator ideal and the nilpotent elements of R form an ideal.

The proof of this lemma, which depends on a deep result of Amitsur on PI-rings, may be found in [2].

Lemma 2. Let R be a ring satisfying the hypothesis of Theorem 1 and having no nonzero divisors of zero; and let (x, y) be an arbitrary ordered pair of elements of R. If p(X) \in XZ[X] is such that xy - yp(x) \in Z, then xy^2 = y^2x or xy = yp(x).

Proof. Suppose that xy^2 \neq y^2x, and write
\[ xy = yp(x) + z, \text{ where } z \in Z; \]
and let \( p_1(X) = XZ[X] \) be such that
\[ x^2y - yp_1(x^2) \in Z. \]
Repeated substitution of (3) in (4) yields \( x(yp(x) + z) - yp_1(x^2) \in Z, \)
\( (yp(x) + z)p(x) + xz - yp_1(x^2) \in Z, \) and finally
\[ y((p(x))^2 - p_1(x^2)) + z(x + p(x)) \in Z. \]
If \( (p(x))^2 - p_1(x^2) \neq 0, \) (5) implies that \( xy = yx, \) contrary to our supposition that \( xy^2 \neq y^2x; \) hence
\[ (p(x))^2 - p_1(x^2) = 0 \text{ and } z(p(x) + x) \in Z, \]
so that \( z = 0 \text{ or } p(x) + x \in Z. \)
But if \( p(x) + x \in Z, \) then (3) yields \( xy - yp(x) = xy - y(p(x) + x) + yx \in Z, \)
implying that \( y \) commutes with \( xy + yx \) and, hence, that \( y^2 \) commutes with \( x; \) therefore \( z = 0 \) and (3) now shows that \( xy = yp(x). \)

Proof of Theorem 1. It will suffice to show that prime rings satisfying the hypothesis of Theorem 1 are commutative (see [2]). Accordingly, let \( R \) be such a prime ring; we first show that \( R \) has no nonzero divisors of zero.
Suppose that \( ab = 0, a \neq 0, \) and \( r \) is an arbitrary element of \( R. \) There exists \( q(X) \in XZ[X] \) for which \( b(ra) - (ra)q(b) \in Z; \) and since \( aq(b) = 0, \) we have \( b(ra) \in Z \) and thus \( sa(bra) = 0 = (bra)s \) for all \( s \in R. \) The primeness of \( R \) now implies that \( bra = 0 \) and hence that \( b = 0. \)
Assume that \( R \) is a noncommutative prime ring satisfying (2). The identity
\[ (xy^2 - y^2x)(yx^2 - x^2y)(xy^2x - yx^2y) = 0 \]
is not satisfied by the ring of 2 \times 2 matrices over any field \( GF(p), \) as may be verified by substituting the matrices \[ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } x \text{ and } y \text{ respectively}; \]
thus, by Lemma 1, \( R \) cannot satisfy (7), and there must exist elements \( a, b \) of \( R \) for which \( ab^2 - b^2a, ba^2 - a^2b, \) and \( ab^2a - ba^2b \) are all nonzero.
If \( p(x) \in \mathbb{Z}[x] \) is such that \( ab = bp(a) \in \mathbb{Z} \), it follows from Lemma 2 that
\[
ab = bp(a).
\]

Now let \( s(x) \in \mathbb{Z}[x] \) satisfy
\[
bp(a) - p(a)s(b) \in \mathbb{Z}
\]
and apply the result of Lemma 2 to the ordered pair \((b, p(a))\). If \((p(a))^2 b = b(p(a))^2\), it follows from (8) that \( a^2 b = a(bp(a)) = b(p(a))^2 = (p(a))^2 b\), so that \( a^2 = (p(a))^2\) and \( a^2\) commutes with \( b\), contrary to the choice of \( a \) and \( b\). Therefore, by Lemma 2, \( bp(a) = p(a)s(b)\), which combines with (8) to give
\[
ab = p(a)s(b).
\]

Now it is immediate from Lemma 2 that \( R \) is an Ore domain and can be embedded in a division ring \( D \). In \( D \), (10) implies that \( b(s(b))^{-1} = a^{-1} p(a) \) commutes with both \( a \) and \( b\); and (8) written in the form \( ab = baa^{-1} p(a) \) shows that \( ab \) and \( ba \) commute, contrary to the original choice of \( a \) and \( b\). This contradiction completes the proof of Theorem 1.

2. Two commutativity theorems. In this section we single out two conditions of type (2) which imply commutativity.

**Theorem 2.** Let \( R \) be a ring such that for every ordered pair \((x, y)\) of elements of \( R \), there exists an integer \( n = n(x, y) \geq 1 \) for which \( xy = yx^n \). Then \( R \) is commutative.

**Lemma 3.** Any ring \( R \) satisfying the hypothesis of Theorem 2 has each of the following properties:

(a) Idempotents of \( R \) are central.

(b) \( R \) is a duo ring (i.e. one-sided ideals are two-sided); moreover \( ab = 0 \) implies \( ba = 0 \), so that there is no distinction between right and left zero divisors.

(c) Commutators in \( R \) are central.

(d) If \( a, b \in R \) are such that \( a(ab - ba) = b(ab - ba) = 0 \), then \( ab - ba = 0 \); similarly, if \( a(ab - ba)x = b(ab - ba)x = 0 \) for some \( x \in R \), then \( (ab - ba)x = 0 \).

**Proof.** (a) If \( x \in R \) and \( e \) is idempotent, there exist positive integers \( m, n \) such that \( e(ex - exe) = (ex - exe) e^m \) and \( e(ex - exe) = (xe - exe) e^n \); hence \( ex - exe = xe - exe = 0 \).

(b) Let \( I \) be a right ideal of \( R \), \( a \in I \) and \( r \in R \); note that since \( ra = ar^n \) for some \( n \geq 1 \), \( ra \in I \). Thus all right ideals are two-sided, and a similar argument holds for left ideals.

Now let \( ab = 0 \). Since \( ba = ab^n \) for some \( n \geq 1 \), \( ba = 0 \) as well.

(c) By Theorem 1 the commutator ideal is nil and, hence, contained in the Jacobson radical \( J(R) \); therefore, it will suffice to show \( J(R) \subseteq \mathbb{Z} \). If we
assume the existence of an element \( a \in J(R) \setminus \mathbb{Z} \), then there is an element \( b \in R \) and integers \( m, n > 1 \) for which \( ab = ba^m \) and \( ba = ab^n \neq ab \). It follows that \( ab = ab^{n-1}a^{m-1} \), and because \( b^{n-1}a^{m-1} \in J(R) \), we now have \( ab = 0 \). Similarly, \( ba = 0 \) and we have a contradiction.

(d) Suppose \( a(ab - ba) = b(ab - ba) = 0 \); in view of (c), \( a^2b = ba^2 \) and \( b^2a = ab^2 \). Suppose \( ab - ba \neq 0 \) and let \( m, n > 1 \) be such that \( ab = ba^m \) and \( ba = ab^n \). Substituting each of these expressions into the other yields \( ab = ab^{n-1}a^{m-1} \) and \( ba = ba^m b^{n-1} \). If \( m \) and \( n \) are both even we thus get \( ab = ba = a^m b^n \); on the other hand, if one of \( n, m \) is odd, we have

\[
ab - ba = a b^{n-1}a^{m-1} - b a^{m-1}b^{n-1} = (ab - ba) a^{m-1} b^{n-1},
\]

which is zero since \( (ab - ba)a = 0 \).

Finally, if \( x \in R \) and \( A \) is the annihilator of \( x \), we get the second statement of (d) by applying the preceding argument to the ring \( R/A \).

**Proof of Theorem 2.** It will suffice to prove commutativity under the additional hypothesis that \( R \) is subdirectly irreducible, in which case (since \( R \) is a duo ring) the set of zero divisors is precisely the annihilator of the unique minimal ideal \( S \) [1, Lemma 3].

The initial step is to show that zero divisors in \( R \) are central. Accordingly, suppose \( a \) is a noncentral zero divisor which fails to commute with some element \( b \in R \); and consider the case where \( b \) is also a zero divisor. Then by (d) of Lemma 2, we have one of \((ab - ba)a \) and \((ab - ba)b\) different from 0 and \((ab - ba)R\) is a nontrivial ideal; therefore if \( 0 \neq s \in S \), there exists an element \( x \in R \) for which \( s = (ab - ba)x \). But \( 0 = as = bs = a(ab - ba)x = b(ab - ba)x \), and from (d) of Lemma 2 we then get \((ab - ba)x = 0\), a contradiction. Now consider the case where \( b \) is not a zero divisor and let \( m, n > 1 \) be such that \( ab = ba^m \) and \( ba = ab^n \). Since \( ab \) is a zero divisor, \( ab \) and \( a \) commute, so that \( a(ab - ba) = (ab - ba)a = 0 \) and \( a^2 \) commutes with \( b \). If \( m \) is odd, repeating some of the computation in Lemma 2(d) shows that \( ab - ba = (ab - ba)a^{m-1}b^{n-1} = 0 \); on the other hand, if \( m \) is even, \( ab = a^m b, a^m = a, \) and \( a^{m-1} \) is a nonzero idempotent. Recalling that any nonzero central idempotent of a subdirectly irreducible ring must be a multiplicative identity element, we get a contradiction of the fact that \( a \) was a zero divisor. Therefore zero divisors of \( R \) are central.

Now suppose that \( R \) is not commutative and \( b \notin \mathbb{Z} \). There then exist \( a \in R \) not commuting with \( b \) and an integer \( j > 1 \) such that \( ba = ab^j \). Since \( a \) cannot be a zero divisor and since \( ab - ba = a(b - b^j) \) is a zero divisor (nilpotent, in fact), \( b - b^j \) must be a zero divisor, hence central. We have now arrived at a contradiction of Herstein’s well-known result that a ring \( R \) is commutative if for each \( x \in R \), there is an integer \( n(x) > 1 \) for which \( x - x^{n(x)} \in \mathbb{Z} \); and our proof is complete.

**Theorem 3.** Let \( R \) be a ring such that for every ordered pair \((x, y)\) of
elements of $R$, there is a polynomial $p(X) \in XZ[X]$ such that $xy = yxp(x)$. Then $R$ is commutative.

**Proof.** Again applying the given condition to $e$, $ex - exe$, and $xe - exe$ shows that idempotents must be central. Also, since $x^2 = x^2p(x)$ for some $p(X) \in XZ[X]$, $R$ is periodic by a result of Chacron [3]; therefore, $R$ is either nil or contains a nonzero idempotent.

Suppose now that $R$ is subdirectly irreducible. If $R$ contains a nonzero idempotent, then it must have an identity; thus, for each $x \in R$ we have $x = xp(x)$, and $R$ is commutative by the major theorem of [5]. On the other hand, if $R$ is nil we have

$$xy = yxp(x) = yxp(x)q(y)p(x) = yxp(x)q(y)p(x)$$

for an appropriate element $r(X) \in XZ[X]$. In particular, $xy = yxyz$ for some element $z_1 \in R$; and, continuing inductively, for each positive integer $n$ we get an element $z_n \in R$ for which $xy = y^nxyz$, so that $xy = 0$ and $R$ is a zero ring. Therefore, if $R$ is subdirectly irreducible, it is commutative; and the proof of Theorem 3 is finished.

The hypothesis of Theorem 3 cannot be weakened to the condition that $xy - yxp(x) \in Z$, as we see by noting that there exist noncommutative rings satisfying the identity $x^2 = 0$. However, it may be of some interest (but not enough to justify including the proof) to note that rings satisfying the weaker hypothesis are polynomial-identity rings—satisfying the identity $[[x, y], z]^2[x, y] = 0$.

**REFERENCES**


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