PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

CYCLOTOMIC SPLITTING FIELDS FOR GROUP CHARACTERS

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ABSTRACT. This paper is concerned with cyclotomic splitting fields for a real-valued irreducible character of a finite group. The fields considered are of the form $Q(\epsilon_m)$, where m is either an odd prime or a power of 2.

Let χ be an irreducible character of G and let ϵ_m be a primitive mth root of unity. A famous theorem of Richard Brauer states that if m is the exponent of G, then $Q(\epsilon_m)$ is a splitting field for G. In a paper where he gives his second proof of this theorem, Brauer states the following proposition without proof [2, Theorem 3]: If χ is a real-valued character of G, then there exists an element of G whose order m is either an odd prime or a power of 2 such that $Q(\epsilon_m)$ splits χ. The examples given below show that this proposition is actually false. One weaker theorem is proved by B. Fein [3]. The Theorem given below is another attempt to substitute for Brauer's proposition.

Let k be a field of characteristic 0. The pair (G, χ) is said to be *k-special* if there exists a normal, cyclic, self-centralizing subgroup A of G and a faithful linear character λ of A such that $\chi = \lambda^G$ and G/A acts on λ as $\mathrm{Gal}(k(\lambda)/k(\chi))$. Many questions on the Schur index reduce to considering such k-special pairs. Basic results on the Schur index can be found in Yamada [4].

Theorem. *Suppose that χ is a real-valued character of G and G contains no elements of order $4n$ with n odd and $n > 1$. Then there exists an integer m dividing the exponent of G such that m is either an odd prime or 4, and such that $Q(\epsilon_m)$ splits χ.*

Proof. To prove the Theorem, it is necessary to show that the Schur index $m_F(\chi)$ equals 1 for some field $F = Q(\epsilon_m)$ as specified above. Since χ is real-valued, then $m_Q(\chi) \leq 2$ by the Brauer-Speiser theorem. By the the Brauer-Witt theorem, it suffices to consider $Q(\chi)$-special pairs (G, χ) where G/A is a 2-group.

If G is a 2-group, then $m_Q(\chi) = 1$ if $\exp(G) = 2$. If $4|\exp(G)$, then m

Received by the editors October 25, 1974.
AMS (MOS) subject classifications (1970). Primary 20C15.
Key words and phrases. Schur index, Brauer-Speiser theorem, Brauer-Witt theorem.
[1]This research was supported by NSF Grant GP-29437.

$= 4$ satisfies the conclusion of the Theorem. For the remainder of the proof, assume that there exists an odd prime q which divides $|G|$. It will be shown that $F = Q(\epsilon_q)$ splits χ.

Assume $m_Q(\chi) = 2$. Let p be a prime such that $m_{Q_p}(\chi) = 2$. Let H be a subgroup of G such that $A \subseteq H$ and H/A acts on λ as $\mathrm{Gal}(Q_p(\lambda)/Q_p(\chi))$. Set $\phi = \lambda^H$. Then $m_{Q_p}(\phi) = m_{Q_p}(\chi) = 2$. Suppose H contains no element of order 4. Then a Sylow 2-subgroup of G is elementary abelian and A has a complement T in H. Thus $\phi(1) = |T|$ and $(\phi, (1_T)^H) = 1$, so $m_{Q_p}(\phi) = 1$, which is a contradiction. Therefore H contains an element of order 4. Let x be an element of order q in A. Since G contains no element of order $4q$, then $x \notin Z(H)$. Since λ is faithful and H/A acts on λ as $\mathrm{Gal}(Q_p(\lambda)/Q_p(\chi))$, then $2 \| Q_p(\chi, \epsilon_q) : Q_p|$. Thus, if $k = Q_p(\epsilon_q)$, then $m_k(\chi) = 1$. Therefore $m_F(\chi) = 1$ for $F = Q(\epsilon_q)$.

Example (1). Define $G = \langle a, b, c, z, x, y, w\rangle$ with the following relations:

$$a^5 = b^{11} = c^{43} = z^2 = x^4 = w^{42} = 1, \qquad y^{10} = z,$$
$$[x, w] = z, \qquad x^{-1}ax = a^2, \qquad y^{-1}by = b^2, \qquad w^{-1}cw = c^3.$$

Then $\exp(G) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 43$. Let $A = \langle a, b, c, z\rangle$ and let λ be a faithful linear character of A. Then $A \lhd G$ and $\chi = \lambda^G$ is a rational-valued irreducible character of G. The p-local Schur indices of χ can be calculated by using either the formula of Berman [1, §4] or Yamada [4, Chapter 4]. The index $m_{Q_p}(\chi) = 2$ exactly when $p = 5, 11, 43$, and ∞. Furthermore, if $m \in \{4, 3, 5, 7, 11, 43\}$, there exists $p \in \{5, 11, 43, \infty\}$ such that $|Q_p(\epsilon_m) : Q_p|$ is odd. Thus $Q(\epsilon_m)$ fails to split χ for each such m. Hence Brauer's proposition is false.

Example (2). Another example shows that if $\exp(G)$ is replaced by $|G|$, then the proposition is still false. Define $G = \langle a, b, c, z, x, y, w\rangle$ with the following relations:

$$a^{17} = b^{31} = c^{103} = z^2 = x^2 = y^2 = 1, \qquad w^2 = z,$$
$$[x, y] = z, \qquad x^{-1}ax = a^{-1}, \qquad y^{-1}by = b^{-1}, \qquad w^{-1}cw = c^{-1}.$$

Then $|G| = 2^4 \cdot 17 \cdot 31 \cdot 103$. Set $A = \langle a, b, c, z\rangle$, λ a faithful character of A, and $\chi = \lambda^G$. Then χ is real-valued and has local Schur index 2 at 17, 31, and 103. Furthermore, if $m \in \{16, 17, 31, 103\}$, there exists $p \in \{17, 31, 103\}$ such that $|Q_p(\epsilon_m) : Q_p|$ is odd. Therefore $Q(\epsilon_m)$ fails to split χ for each m.

The following result shows that this situation cannot happen if χ is rational-valued.

Proposition. *Let χ be an irreducible character of G such that $Q(\chi)$ is an extension of Q of odd degree. If $|G| = 2^c n$, n odd, then $Q(\epsilon_{2^c})$ splits χ.*

Proof. By the Brauer-Speiser theorem, $m_Q(\chi) \leq 2$. By the Brauer-Witt theorem, it suffices to consider $Q(\chi)$-special pairs (G, χ) where G/A is a 2-group. Since $Q(\chi)/Q$ has odd degree, G/A is isomorphic to a Sylow 2-subgroup of $\mathrm{Gal}(Q(\lambda)/Q)$.

Suppose $m_Q(\chi) = 2$. Then χ cannot be linear, so $G \neq A$. Hence $2 \mid |G:A|$. Let T be a Sylow 2-subgroup of G. If $A \cap T = \langle 1 \rangle$, then $\chi(1) = |G:A| = |T|$ and $(\chi, (1_T)^G) = 1$. In that case, $m_Q(\chi) = 1$, which is a contradiction. Hence $2 \mid |A|$ so $4 \mid |G|$ and $c \geq 2$. Thus $2 \mid |Q_p(\epsilon_2 c): Q_p|$ for $p = 2, \infty$. In particular, $Q_2(\epsilon_2 c)$ and $Q_\infty(\epsilon_2 c)$ each split χ.

Let p be an odd prime with $p - 1 = 2^a b$, b odd. Suppose $m_{Q_p}(\chi) = 2$. Then $p \mid |A|$. Since λ is faithful and G/A is isomorphic to a Sylow 2-subgroup of $\mathrm{Gal}(Q(\lambda)/Q)$, $2^a \mid |G:A|$. Therefore, $c \geq a + 1$. Hence, $2 \mid |Q_p(\epsilon_2 c): Q_p|$ so $Q_p(\epsilon_2 c)$ splits χ.

Therefore $Q(\epsilon_2 c)$ splits χ.

REFERENCES

1. S. D. Berman, *Representations of finite groups over an arbitrary field and over rings of integers*, Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), 69–132; English transl., Amer. Math. Soc. Transl. (2) 64 (1967), 147–215. MR 33 #5747.

2. R. Brauer, *Applications of induced characters*, Amer. J. Math. 69 (1947), 709–716. MR 9, 268.

3. B. Fein, *Realizability of representations in cyclotomic fields*, Proc. Amer. Math. Soc. 38 (1973), 40–42. MR 47 #3503.

4. T. Yamada, *The Schur subgroup of the Brauer group*, Lecture Notes in Math., vol. 397, Springer-Verlag, Berlin and New York, 1974.

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