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ON THE RADIUS OF STARLIKENESS OF $(zf)'$ FOR f UNIVALENT

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ABSTRACT. Let S be the standard class of normalized univalent functions. For a given function f, $f(z) = z + a_2z^2 + \cdots$, regular for $|z| < 1$, let $r(f)$ be the radius of starlikeness of f. In 1947, R. M. Robinson considered the combination $g_f(z) = (zf)'/2$ for $f \in S$. He found a lower bound of .38 for $r(g_f)$ for all $f \in S$. He noted that the standard Koebe function k, $k(z) = z(1-z)^2$, has its $r(g_k)$ equal to $\frac{1}{2}$. A question that has been asked since Robinson's paper is whether $\frac{1}{2}$ is the minimum $r(g_f)$ for all f in S. It is shown here that this is not the case by giving examples of functions f whose $r(g_f)$ is less than $\frac{1}{2}$.

Introduction. Let $U = \{z\colon |z| < 1\}$. For a given f, $f(z) = z + a_2z^2 + \cdots$, regular in U, let $r_0(f)$ be the radius of univalence of f and $r_1(f)$ the radius of starlikeness of f (see Hayman [4] for definitions). Let S be the class of normalized functions f, $f(z) = z + \cdots$, regular and univalent in U. Let S^* be the standard subclass of S of starlike functions. In 1947, R. M. Robinson [10] considered the combination $g_f(z) = (zf)'/2$ for $f \in S$. He found a lower bound of .38 for $r_1(g_f)$ for all $f \in S$. He noted that $g_f'(z) \neq 0$ for all z, $|z| < \frac{1}{2}$. He also noted that the standard Koebe function k, $k(z) = z/(1-z)^2$, has $g_k'(-\frac{1}{2}) = 0$ and that $r_0(g_k) = r_1(g_k)$, i.e., $g_k = (zk)'/2$ is a starlike function and, hence, univalent for z, $|z| < \frac{1}{2}$, while it is not univalent and, hence, not starlike in any disk $|z| < r$ where $r > \frac{1}{2}$.

Let K be the standard subclass of S of functions that are close to convex in U (see Goluzin [3] for definition). In 1966, A. E. Livingston [7] showed that if $f \in S^*$, K, then, $g_f(|z| \leq \frac{1}{2})$ is starlike, close to convex, respectively. The Koebe function shows the $\frac{1}{2}$ to be sharp in both cases. These results have been generalized by Bernardi [1], [2] and others [6], [9]. In each case the Koebe function or its appropriate generalization has given the sharpness results. The natural question that has been asked is whether the $\frac{1}{2}$ obtained from the Koebe function is the smallest radius of starlikeness and, hence, the smallest radius of univalence for all functions $g_f = (zf)'/2$, $f \in S$. This author shows that $\frac{1}{2} = r_1(g_k)$ is not the smallest $r_1(g_f)$ for all $f \in S$. We give two examples of functions f whose $r_1(g_f)$ is less

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than $\frac{1}{2}$. The first example is a nonsymmetric function f_1 having $r_1(g_{f_1})$ less than 0.445. We also give as a second example a circularly symmetric function h for which $r_1(g_h)$ is less than 0.493. Therefore the Koebe function does not give the minimum $r_1(g_f)$ for all f in the class of symmetric, or even circularly symmetric, functions in S. The question as to whether $\frac{1}{2} = r_0(g_k)$ is the minimum radius of univalence of g_f for all f in S is still open.

Example 1. Let a be such that $0 < a < 1$. Consider the function $f_1 \in S$ such that the complement of $f_1(U)$ is the union of a radial ray from the point $A < 0$ to $-\infty$ and the line segment connecting A to a point B with the angle opening at A being $a\pi$. Because starlikeness is independent of magnification, we may assume $A = -1$. Then an explicit formula for the function f_1 can be found by translating A to the origin and then taking the logarithm. Let $f_2(z) = f_1(z) - A$. Then $\log f_2(U)$ is the strip of width 2π symmetric about the real axis and minus a ray that is parallel to and a units away from the real axis. The Schwartz-Christoffel formula enables us to obtain $f_2(z)$ (up to rotation and translation) from the equation

$$\log f_2(z) = \int_0^z \frac{(1 - e^{id}w)\,dw}{(1 - w)(1 - e^{ib}w)(1 - e^{ic}w)} \tag{1}$$

with $0 < c \le d \le b < 2\pi$. Then $f_2(z) = (1 - e^{ib}z)^C(1 - e^{ic}z)^D/(1 - z)^{C+D}$ where C and D are the constants obtained from the partial fraction decomposition of the integrand in (1). To assure that f_2 maps onto the required domain we let $C = a$ and choose $C + D = 2$ with $b = [2\pi - c(2 - a)]/a$. Hence

$$f_2(z) = (1 - e^{ib}z)^a(1 - e^{ic}z)^{2-a}/(1 - z)^2$$

Now, we have

$$\begin{aligned} 2g_{f_1}(z) = F(z) = f_1(z) + zf_1'(z) &= f_2(z) + zf_2'(z) - 1 \\ &= f_2(z)[1 + zf_2'(z)/f_2(z)] - 1. \end{aligned}$$

So that for $D = 2 + (a - 2)e^{ic} - ae^{ib}$ and $\bar{D}$, the complex conjugate of D, we obtain

$$F(z) = \left(\frac{1 - e^{ic}z}{1 - e^{ib}z}\right)^{1-a} \frac{(1 - z)(1 - e^{ic}z)(1 - e^{ib}z) + Dz + \bar{D}e^{i(c+b)}z^2}{(1 - z)^3} - 1. \tag{2}$$

The form given adapts readily to computer language. The Wang 2200B was used to compute $F(z)$ for values on the circle $|z| = \frac{1}{2}$. By letting a vary, the largest decrease in the argument of $F(z) = F(z, c, a)$ was obtained by letting $a = 0.9$ and $c = 1.3$, i.e., $\arg F(e^{i\theta}/2, 1.3, 0.9)$ decreases from 144.4° at $\theta = 80.2^\circ$ to 139.1° at $\theta = 130.6^\circ$.

Because of the caution used in interpreting computer results, we offer here examples which, although tedious, can be computed by hand. However, to obtain a higher degree of accuracy, the involved computations have been performed by an HP45 hand calculator carrying the figures out to 9 decimal places, then rounding off to 5 decimal places to be written out in this paper. We note here that a slightly different form for $F(z, c, a)$ was used in the hand calculations and the final results agreed with the output of the computer to at least 7 decimal places. This was also true for the second example given in this paper. A simple computation shows that

$$F(z, c, a) = \left(\frac{1 - e^{ic}z}{1 - e^{ib}z}\right)^{1-a} \frac{1 + D_1 z + E_1 z^2 - e^{i[c+b]}z^3}{(1 - z)^3} - 1$$

with $D_1 = 1 + (a - 3)e^{ic} - (a + 1)e^{ib}$ and $E_1 = 3e^{i(c+b)} + (a - 1)e^{ib} + (1 - a)e^{ic}$. Let $a = 0.9$, $c = \pi/2$, $s = \sin(\pi/9)$ and $t = \cos(\pi/9)$. Then $D_1 = 1 - 1.9s - i(2.1 - 1.9t)$, $E_1 = 3t - 0.1s + i(3s + 0.1t + 0.1)$ and $b = 29\pi/18$. Also $e^{i29\pi/18} = -ie^{i\pi/9}$. Thus

$$F(z, \pi/2, 0.9) = \left(\frac{1 - iz}{1 + ie^{i\pi/9}z}\right)^{0.1} \frac{1 + D_1 z + E_1 z^2 - e^{i\pi/9}z^3}{(1 - z)^3} - 1. \tag{3}$$

Now, using (3) we get

$$F(e^{i\pi/2}/2, \pi/2, 0.9)$$

$$= \left(\frac{3}{2 - t - si}\right)^{0.1} \frac{8 + 4D_1 i - 2E_1 + ie^{i\pi/9}}{(2 - i)^3} - 1 = \left(\frac{3}{2 - t - si}\right)^{0.1}$$

$$\cdot \frac{8 + 4(2.1 - 1.9t) + 4(1 - 1.9s)i - 6t + 0.2s - (6s + 0.2t + 0.2)i - s + ti}{2 - 11i} - 1$$

$$= \frac{3^{0.1}[16.4 - 13.6t - 0.8s + (3.8 + 0.8t - 13.6s)i](2 + 11i)}{(2 - t - si)^{0.1}(2 - 11i)(2 + 11i)} - 1$$

$$= \frac{3^{0.1}[-9 - 36t + 148s + (188 - 148t - 36s)i]}{(2 - t - si)^{0.1}125} - 1$$

$$= \frac{7.79005 + 36.61277i}{(1.01037 - 0.03154i)125/3^{0.1}} - 1$$

$$= \frac{-105.36633 + 40.14473i}{113.15638 - 3.53196i}$$

$$= \frac{112.75488e^{i159.14309^\circ}}{113.21149e^{-1.78780^\circ}}$$

$$= .99597e^{i160.93088^\circ}.$$

While

$$F(e^{i3\pi/4}/2,\, \pi/2,\, 0.9)$$

$$= \left(\frac{4 - \sqrt{2}(i-1)i}{4 + i\sqrt{2}(i-1)e^{\pi/2}}\right)^{0.1} \frac{1 + \sqrt{2}(i-1)D_1/4 - iE_1/4 - \sqrt{2}(1+i)(t+is)/16}{(1-\sqrt{2}(i-1)/4)^3} - 1$$

$$= \left(\frac{4 + \sqrt{2} + \sqrt{2}i}{4 - \sqrt{2}(t-s) - \sqrt{2}(t+s)i}\right)^{0.1} \cdot \left[\frac{4[16.4 + 4.4\sqrt{2} + (0.4 - 8.6\sqrt{2})t + (12 + 8.6\sqrt{2})s]}{64 + 44\sqrt{2} - (48 + 52\sqrt{2})i}\right.$$

$$\left. + \frac{4i[12.4\sqrt{2} - (12 + 8.6\sqrt{2})t + (0.4 - 8.6\sqrt{2})s]}{64 + 44\sqrt{2} - (48 + 52\sqrt{2})i}\right] - 1$$

$$= \frac{(1.18753 + 0.03035i)}{(1.13632 - 0.05931i)} \frac{4(19.83362 - 9.19175i)}{(126.22540 - 121.53911i)} - 1$$

$$= \frac{95.32796 - 41.25437i}{136.22345 - 145.59414} - 1 = \frac{-40.89549 + 104.33979i}{136.22345 - 145.59414i}$$

$$= \frac{112.06799e^{i111.40246^\circ}}{199.38526e^{-i46.904444^\circ}} = .56207e^{i158.30690^\circ}.$$

Since $\arg F(e^{i3\pi/4}, \pi/2, 0.9) < \arg F(e^{i\pi/2}, \pi/2, 0.9)$, $F(|z| < \tfrac{1}{2})$ is not a starlike domain for $c = \pi/2$ and $a = 0.9$. An examination of $F(z, c, a)$ by the computer to find an approximation to the radius of starlikeness of F showed that the $\arg F(.445e^{i\theta}, 1.3, 0.9)$ was a decreasing function of θ at $\theta = 1.825$. Thus the minimum $r_1(g_f)$ for all $f \in S$ can be no larger than 0.445.

Example 2. In this section we give an example of a circularly symmetric function h (see Jenkins [5] for definitions) for which the radius of starlikeness of $(zh)'/2$ is less than 0.494. We use the functions h_a, $0 < a < 1$, which have been discussed by T. Suffridge [11], and have been used by E. Netanyahu [8] to solve an important extremal problem. The details in the verification of the properties of these functions can be found in [11]. Let h_a be defined by

$$G_a(z) = \frac{zh_a'(z)}{h_a(z)} = \frac{1 + 2bz + z^2}{(1 + 2cz + z^2)^{1/2}(1 - z)}$$

where $b = 2a - 1$, $c = 2a^2 - 1$, $z \in U$, $0 < a < 1$. Since $\partial/\partial\theta \log h_a(e^{i\theta}) = iG(e^{i\theta})$ (any branch of $\log h_a(e^{i\theta})$, $0 < \theta < 2\pi$, can be chosen), it follows from the boundary behavior of $zh_a'(z)/h_a(z)$ that h_a maps U onto the complex plane minus a set $\{w\colon w \leq -(1+a)^{-2}$ or $|w| = (1+a)^{-2}$ and $\pi - \psi \leq \arg w \leq \pi + \psi\}$ $(0 < \psi < \pi)$, where $\psi = \pi - 2\cos^{-1}[(1-a)/(1+a)]$. Then for $2F_a(z) = [zh_a(z)]'$, we have

$$H_a(z) = zF_a'(z)/F_a(z) = G_a(z) + zG_a'(z)/[G_a(z) + 1].$$

Thus, for $B = 1 + 2bz + z^2$ and $C = 1 + 2cz + z^2$,

$$(4) \quad H_a(z) = \frac{B}{(1-z)C^{1/2}} + \frac{z[2(1-z)(b+z)C + BC - (1-z)(c+z)B]}{C(1-z)[B + (1-z)C^{1/2}]}.$$

The computer showed that if $a_0 = 0.695$, then $\operatorname{Re}\{H_{a_0}(z)\} < 0$ for $z = e^{i\theta}/2$, $1.73 \le \theta \le 1.96$ and $\operatorname{Re}\{H_{a_0}(e^{i\,1.84}/2)\} < -0.0233$. Since the negative variation from zero is so small, we offer here an example that can be calculated by hand. An algebraic manipulation gives

$$H_a(z) = \frac{B}{(1-z)C^{1/2}} + \frac{z[(1+2b-c)+(1+2bc+3c)z+(1+2bc+3c)z^2+(1+2b-c)z^3]}{(1-z)C[B+C^{1/2}(1-z)]}.$$

Let $a_1 = 0.58$ and $z_1 = i/2$. Hence, $b = 0.16$ and $c = -0.3272$. Then

$$H_{a_1}(z_1) = \frac{0.75+0.16i}{(1-0.5i)(0.75-0.3272i)^{1/2}}$$
$$+\frac{i}{2}\,\frac{1.6472 - 0.04315i + 0.02158 - 0.20590i}{(1-0.5i)(0.75-0.3272i)[(0.75+0.16i)+(0.75-0.3272i)^{1/2}(1-0.5i)]}$$
$$= \frac{0.75+0.16i}{(1-0.5i)(0.88551-0.18475i)}$$
$$+\frac{i}{2}\,\frac{1.6687-0.24905i}{(0.5864-0.7022i)(0.75+0.16i+0.79314-0.62751i)}$$
$$= \frac{(0.75+0.16i)(0.79314+0.6275i)}{1.02283} + \frac{i}{2}\,\frac{(1.66878-0.24905i)}{(0.57661-1.35774i)}$$
$$= \frac{0.49445}{1.02283} - \frac{2.12215}{4.35186} + \text{imaginary part}$$
$$= -.00423 + \text{imaginary part}.$$

For completeness we give the following details:

$$(0.75-0.3272i)^{1/2} = [(0.75)^2 + (-0.3272)^2]^{1/4}$$
$$\cdot\left[\cos\frac{\tan^{-1}(-3272/7500)}{2} + i\sin\frac{\tan^{-1}(-3272/7500)}{2}\right]$$
$$= 0.90458[\cos(-11.78502) + i\sin(-11.78502)]$$
$$= 0.88551 - 0.18475i.$$

An examination of H_a by the computer to find an approximate minimum radius of starlikeness for F_a for all a showed that $\operatorname{Re}\{H_a(re^{i\theta})\} < 0$ for $a = 0.695$, $\theta = 1.824$ and $r = .4930$. Thus the minimum radius of starlikeness for $(zf)'/2$ for all circularly symmetric functions $f \in S$ is no larger than 0.493.

Remark. Upon further investigation of the examples that the author found of functions f such that the radius of starlikeness of $(zf)'/2$ was less than $1/2$, the radius of close to convexity of $(zf)'/2$ for all the examples found was still at least $1/2$.

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